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DEFORMATIONS OF CLOSED SPACE CURVES

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1. Introduction

In this note we will be interested in closed space curves, that is C^k $(k \geq 2)$ immersions of S^1 into R^3. We say a space curve is *non-degenerate* if the square of the curvature is never zero. This non-degeneracy condition is the classical hypothesis used to insure the existence of the moving Frenet frame along the curve. The question we would like to ask is the following one. Given any two closed non-degenerate space curves, when are they homotopic through a homotopy composed entirely of closed non-degenerate space curves? More precisely we want to study the space N of non-degenerate closed space curves, considered as a subspace of $C^k(S^1, R^3)$, the C^k maps from S^1 into R^3, with the C^k-topology $(k \geq 2)$ [4]. We ask: what are the arc components of N? A continuous path in N will be called a *non-degenerate homotopy*.

It will be convenient to make all homotopies *based*. To make this specific let us first define the Frenet frame for $\gamma(s) \in N$. This is done by fixing the sign of the curvature to be positive, and letting the *principal normal* $t_2(s)$ be defined by $dt_1(s)/ds = k(s)t_2(s)$, where s is arc-length parameter, $t_1(s) = d\gamma(s)/ds$ is the *unit tangent* vector, and $k(s)$ is the curvature of the given space curve γ. One then defines the *binormal vector* $t_3(s)$ by the formula $t_3(s) = t_1(s) \times t_2(s)$. Now let us fix a base point $\theta_0 \in S^1$, and let

$$N_0 = \{\gamma \in N \mid \gamma(\theta_0) = 0,\ t_1(\theta_0) = e_1,\ t_2(\theta_0) = e_2,\ t_3(\theta_0) = e_3\},$$

where (t_1, t_2, t_3) is the Frenet frame of γ, and the e_i are the unit coordinate vectors of R^3 (i.e., $e_1 = (1, 0, 0)$ etc.). An element of N_0 will be called a *based non-degenerate* curve, and a continuous path in N_0 a *based non-degenerate homotopy*. By using rotations and translations the arc components of N are determined by those of N_0, because the group of rigid motions is connected. Given any $\gamma \in N_0$, we define $F(\gamma): S^1 \to SO(3)$, by associating to each point of γ, its Frenet frame, where $SO(3)$ is the special orthogonal group. We see $F(\gamma)$ is of class C^{k-2}, and $F(\gamma)(\theta_0) = f_0 = (e_1, e_2, e_3) \in SO(3)$. Our main result is the following theorem.

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