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SIMPLE CLOSED GEODESICS ON CONVEX SURFACES

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Abstract

A geodesic is said to be simple if it does not have any self-intersection point. It will be shown that the shortest closed geodesic is simple on any smooth Riemannian 2-sphere of nonnegative curvature.

We will also derive various estimates for lengths of simple closed geodesics, in terms of the diameter D, total area A, and curvature K of a given surface M^2. In particular, if we let L be the length of the longest simple closed geodesic on a smooth Riemannian sphere of curvature $0 \le K \le 1$, then $2D \le L \le A/2$. Furthermore, equality $L = A/2$ holds if and only if M^2 is isometric to the unit sphere.

Finally, if M^2 is a Riemannian sphere with nonnegative curvature, then we find that the isoperimetric inequality $A \le 8D^2/\pi$ is useful.

Introduction

The purpose of this note is to study simple closed geodesics on compact oriented convex surfaces. A geodesic γ is said to be simple if γ has no self-intersections. In what follows, all geodesics are assumed to be nontrivial. Hence, any point curve will not be counted as a closed geodesic. If a Riemannian surface M^2 is homeomorphic to the two-sphere S^2 and if M^2 has nonnegative sectional curvature, then M^2 is called a convex surface.

First, we would like to find out which closed geodesics are simple on a given surface. The following theorem gives a partial answer.

Theorem D. *If g is a C^3 smooth metric on a two-sphere S^2 with nonnegative curvature, then any nontrivial closed geodesic of the shortest length is simple.*

In Theorem D, we only consider the C^3 smooth metric g, since there are examples of nonsmooth metrics on a two-sphere S^2 in which the shortest geodesics are not simple. For instance, the bi-equilateral triangle (two

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