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NODAL SETS OF EIGENFUNCTIONS ON RIEMANN SURFACES

RUI-TAO DONG

Abstract

On an n-dimensional smooth Riemannian manifold, the nodal set of an eigenfunction is its zero set. It has been a longstanding problem to estimate the $(n-1)$-dimensional Hausdorff measure of the nodal set in terms of the corresponding eigenvalue and the geometry of the manifold. In this paper, we give an upper bound on the length of the nodal set on a Riemann surface.

Defining the vanishing order of an eigenfunction at a point to be the order of the first nonzero term in its Taylor expansion at that point, we also give an upper bound on the sum of the vanishing orders over the points on the Riemann surface, where the eigenfunction and its gradient both vanish. This result sharpens a similar result by Donnelley and Fefferman.

1. Introduction

Let (M^n, g) be a connected, smooth, compact Riemannian manifold without boundary. Suppose that Δ is the Laplace-Beltrami operator on (M^n, g), and u is a real eigenfunction with corresponding eigenvalue λ, i.e., $\Delta u = -\lambda u$. The nodal set $\mathscr{N}$ of u is defined to be the set of points $x \in M$ where $u(x) = 0$.

Denote D to be the diameter of the manifold, and H to be the upper bound of the absolute value of the sectional curvature.

It is clear that outside of the singular set $\mathscr{S} = \{x \mid u(x) = 0,\ \nabla u(x) = 0\}$, $\mathscr{N}$ is a regular $(n-1)$-dimensional submanifold of M.

Yau conjectured in Problem 73 of [14] that

$$c_1\sqrt{\lambda} \le \mathscr{H}^{n-1}(\mathscr{N}) \le c_2\sqrt{\lambda},$$

and that the constants c_1 and c_2 depend only on the geometry of the manifold. Here $\mathscr{H}^{n-1}(\mathscr{N})$ is the $(n-1)$-dimensional Hausdorff measure of $\mathscr{N}$.

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