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REGULARITY OF ISOMETRIC IMMERSIONS OF POSITIVELY CURVED RIEMANNIAN MANIFOLDS AND ITS ANALOGY WITH CR GEOMETRY

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Abstract

Let M be an n-dimensional Riemannian manifold and F be an isometric immersion of M into $\mathbf{R}^{n+1}$. It is shown that under certain conditions on the sign of principal curvatures of $F(M)$, F satisfies an over-determined system of elliptic partial differential equations after one adds the scalar curvature equation. As a corollary, if M is an analytic manifold of positive sectional curvature, F is analytic and uniquely determined by $F(P)$ and $dF(P)$ at a reference point P of M. An analogous problem in CR geometry is proposed.

0. Introduction and statement of the main results

We are concerned in this paper with the regularity and the uniqueness of isometric immersions of n-dimensional Riemannian manifolds into $\mathbf{R}^{n+1}$. We deal with analytic (C^ω) manifolds. However, one can get a C^∞ version of this paper by replacing every C^ω by C^∞. Consider first the following well-known fact: If M is a C^ω connected Riemannian manifold and F is a continuously differentiable isometry of M onto another C^ω Riemannian manifold $\tilde{M}$, then F is C^ω. Moreover, if O is a point of M, then F is uniquely determined by $F(O)$ and the first partial derivatives of F at O. The reason is that locally F can be expressed as a linear mapping between the normal coordinates of M and $\tilde{M}$ near O and $F(O)$, respectively. Analyticity and uniqueness with respect to the initial data at one point follow from the viewpoint of the local equivalence problem also under the assumption $F \in C^2$ (cf. [2] and [4]). Our question is whether one can remove the hypothesis of analyticity of $\tilde{M}$ when $\tilde{M}$ is a hypersurface in a Euclidean space; namely,

Question 1. Let M be an n-dimensional C^ω Riemannian manifold and $F = (f^1, \cdots, f^{n+1})$ be a C^k, $k \gg 0$, isometric immersion of M into $\mathbf{R}^{n+1}$. Then will F be C^ω? And will F be uniquely determined by $F(O)$ and the first partial derivatives of F at a point?

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