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CONFORMAL DEFORMATION OF METRICS ON S^2

SUN-YUNG A. CHANG & PAUL C. YANG

1. Introduction

On the sphere $S^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\}$ with its canonical metric $g_0 = \sum_{i=1}^3 dx_i^2$ the problem of conformal deformation of metric is to find conditions on the function $K(x)$ so that $K(x)$ is the Gauss curvature of a conformally related metric $g = e^{2u}g_0$. In terms of the Laplacian in the canonical metric this is expressed by the differential equation

$$\Delta u + Ke^{2u} = 1. \tag{1.1}$$

This equation has a solution when K is an even function; in this case Moser [13] showed that there is in fact an even function u solving the differential equation. Moser's approach was to maximize a functional within the class of even functions in H^1. The crucial ingredient for proof of convergence is a sharp version of Trudinger's inequality [12]: for an even $\mathscr{C}^1$ function u with $\int_{S^2} u = 0$, and $\int_{S^2} |\nabla u|^2 = 1$, we have $\int_{S^2} \exp(8\pi u^2) \leqslant c_0$ where c_0 is a universal constant. Without the evenness condition the inequality is weakened to $\int_{S^2} \exp(4\pi u^2) \leqslant c_0'$. Kazdan and Warner gave in [9] the necessary condition

$$\int_{S^2} (\nabla K, \nabla x_i) e^{2u} = 0, \qquad i = 1, 2, 3,$$

where x_i is any of the ambient coordinate functions on S^2.

In our previous article [4] we have obtained the following

Theorem. *On S^2, let $K > 0$ be a smooth function with nondegenerate critical points, and in addition $\Delta K(Q) \neq 0$ where Q is any critical point. Suppose there are at least two local maxima and that at all saddle points of K, $\Delta K(Q) > 0$. Then K admits a solution to equation* (1.1).

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