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A CONSTRUCTION OF STABLE BUNDLES ON AN ALGEBRAIC SURFACE

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1. Let X be a smooth projective algebraic surface over $\mathbf{C}$ and let H be an ample divisor on X. We recall that a bundle $\mathscr{E}$ of rank two and $c_1(\mathscr{E}) = 0$ is H-stable (in the sense of Mumford-Takemoto) if whenever $\mathscr{L}$ is a line bundle on X which admits a nonzero map to $\mathscr{E}$, then we have $(c_1(\mathscr{L}) \cdot H) < 0$. In this paper, we will consider the problem of constructing stable bundles $\mathscr{E}$ on X of rank two with $c_1(\mathscr{E}) = 0$ and $c_2(\mathscr{E})$ a prescribed number. From work of Donaldson [1], this question is a special case of the following: When does a principal SU(2) bundle on a four dimensional Riemannian manifold admit an irreducible self dual connection? In this guise, the problem has been studied by Taubes [4]. There has also been some work on higher dimensional manifolds by Uhlenbeck and Yau. The basic goal is to give conditions on the topology of X so that stable bundles $\mathscr{E}$ of the type considered exist with $c_2(\mathscr{E})$ a given integer. The topological invariant of interest here is $h^0(X, \mathscr{O}(K))$, the number of holomorphic two forms on X. Throughout the paper, we will use h^0 as an abbreviation for $h^0(X, \mathscr{O}(K))$. $[r]$ is the greatest integer in r.

Theorem 1.1. *If $n \geqslant 4([h^0/2] + 1)$, then there is an H-stable bundle $\mathscr{E}$ on X of rank two with $c_1(\mathscr{E}) = 0$ and $c_2(\mathscr{E}) = n$.*

Theorem 1.2. *If $h^0 > 1000$ and $n > (3/2)h^0 + 6$, then there is an H-stable bundle $\mathscr{E}$ on X of rank two with $c_1(\mathscr{E}) = 0$ and $c_2(\mathscr{E}) = n$.*

We note that Taubes constructs bundles of the above type for $n \geqslant (8/3)h^0 + 2$. Our methods are modeled on Taubes' methods, namely both methods are degeneration theoretic. My main motivation for this paper was to see Taubes' argument is an algebro-geometric setting. Actually, the argument we will use is somewhat different than Taubes'.

One's first idea in attacking this problem is to construct a torsion free coherent H-stable sheaf $\mathscr{F}$ on X and to prove that $\mathscr{F}$ can be deformed to a locally free sheaf. However, we have adopted a different but related approach

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