J. DIFFERENTIAL GEOMETRY
24 (1986) 221–263

SUB-RIEMANNIAN GEOMETRY

ROBERT S. STRICHARTZ

1. Introduction

By *sub-Riemannian geometry* we mean the study of a smooth manifold M equipped with a smoothly varying positive definite quadratic form on a subbundle S (distribution) of the tangent bundle TM, where S is assumed to be *bracket generating* (sections of S together with all brackets generate TM as a module over the functions on M), and the resulting geometric structures that arise in analogy with Riemannian geometry. This is a subject which has been studied by a number of different investigators, more or less independently, from a number of different viewpoints under a number of different names (*singular Riemannian geometry* and *Carnot-Carathéorody metric* are most commonly used).

In this paper we attempt to give a coherent introduction to the subject, taking the point of view that the subject is a variant of Riemannian geometry. The main topic is the study of geodesics. The quantitative structure of a sub-Riemannian manifold is easily seen to be equivalent to giving a contravariant metric tensor field ($g^{jk}(x)$ in local coordinates) mapping T^*M to TM which is nonnegative definite and has a null-space N equal to the annihilator of S in T^*M. We call this a *sub-Riemannian metric*, and in terms of it we can define the length of any piecewise smooth curve which is tangent to S (we will call such curves *lengthy*). By a well-known theorem of Chow (see also Carathéodory [5]) it is possible to connect any two points by such a curve if the manifold is connected (which we will always assume, for simplicity), and so we can endow M with a metric d defined to be the infimum of the lengths of all lengthy curves joining two points. On the other hand, from the sub-Riemannian metric we can write down a system of Hamilton-Jacobi equations on T^*M, and any solution is called a *geodesic*. The two most basic equations

Received October 28, 1985 and, in revised form, June 27, 1986. Research was supported in part by National Science Foundation Grant DMS-8401354.