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MULTIPOLE POTENTIALS FOR $SU(n)$ and $SO(n)$

WOODY LICHTENSTEIN

1. It is of physical interest to understand the symmetry properties of combinations of objects with known symmetries. A well-known elementary example is the Maxwell-Sylvester analysis of electric potentials generated by finite configurations of point charges [1]. In Euclidean $\mathbf{R}^3$ with coordinates x, y, z and $r = (x^2 + y^2 + z^2)^{1/2}$ one can choose units so that the potential from a single point charge at the origin is $1/r$. For a dipole centered at 0 with axis of length $\underline{a}$ aligned along the x-axis, the potential at large distances from the origin is well-approximated by $a(\partial/\partial x)(1/r)$. Similarly a configuration of 4 charges

$$\begin{array}{cc} +\cdot & \cdot - \\ \cdot & \cdot \\ - & + \end{array}$$

FIG. 1

as in Fig. 1 generates a quadrupole potential which is a multiple of $\partial^2/\partial y \partial x(1/r)$. In general, associated to any polynomial p there is a multipole potential $M(p) = \partial_p(1/r)$ where ∂_p is the constant coefficient differential operator corresponding to p via the Euclidean metric. Since $1/r$ is harmonic, away from its singularity at 0, $M(p) = 0$ if p is a multiple of r^2, and $M(p)$ is always harmonic. In addition, every polynomial may be written as $p = h + qr^2$ with h harmonic [6] so that M may be viewed as a mapping (hereafter referred to as the multipole mapping) from harmonic polynomials to singular harmonic functions.

In order to study objects with more complicated symmetry than the spherically symmetric point charge, one can replace Euclidean $\mathbf{R}^3$ with the Lie algebra $\mathfrak{g}$ of a compact simple Lie group G. Then G acts on $\mathfrak{g}$ via the adjoint representation, preserving the positive definite metric B, where $-B$ is the Killing form. Orbits of maximal dimension will be said to be regular; all other orbits will be said to be singular. There is an invariant polynomial Q on $\mathfrak{g}$ (for $\mathfrak{su}(n)$, Q is just the discriminant of the characteristic polynomial) which

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