J. DIFFERENTIAL GEOMETRY
15 (1980) 595–603

CONSERVATION LAWS AND DIFFERENTIAL CONCOMITANTS

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1. Introduction

The notion of a conservation law on a manifold has appeared previously in papers by H. Osborn, P. R. Eiseman, D. E. Blair, and the author (see [1], [3], and [7]). In the first three sections of this paper the origin of the conservation law problem for manifolds and some earlier results are reviewed. The last section extends results obtained in [3].

In order to formulate a definition of a conservation law on a manifold let us recall the notion of a conservation law in physics. A conservation law is an equation in the form

$$\frac{\partial u}{\partial r} + \sum_{j=1}^{3} \frac{\partial v_j}{\partial x_j} = 0,$$

which simply expresses the fact that the quantity of u contained in a domain D of (x_1, x_2, x_3) space changes at a rate equal to the flux of the vector (v_1, v_2, v_3) into D; i.e.,

$$\frac{d}{dt} \int\int\int_D u \, dx_1 dx_2 dx_3 = \int_\partial\int_D \vec{v} \cdot \vec{n} \, dS.$$

Conservation law form may also sometimes be obtained from a system

$$V_t + AV_x = 0, \tag{1.1}$$

where V is a column vector of n unknown functions, A is a square matrix depending on V, t, and x, and the subscripts t and x denote partial differentiations. If we can write $AV_x = W_x$, then

$$V_t + W_x = 0 \tag{1.2}$$

is a system of conservation laws. These examples then lead (see [8]) to the definition of a conservation law on a manifold. A differential 1-form φ on a manifold M is a conservation law for a linear operator, on 1-forms, if both φ and $\underline{h}\varphi$ are exact. A general problem is then to determine all conservation

Communicated by A. Nijenhuis, October 17, 1978, and, in revised form, March 21, 1979.