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KATO'S INEQUALITY AND THE SPECTRAL DISTRIBUTION OF LAPLACIANS ON COMPACT RIEMANNIAN MANIFOLDS

H. HESS, R. SCHRADER & D. A. UHLENBROCK

1. Introduction

In this note we study Bochner Laplacians D_∇ given by connections ∇ on a smooth hermitian vector bundle V over a compact Riemannian manifold M (without boundary for simplicity). We will compare the spectrum of D_∇ with that of the Laplace-Beltrami operator Δ_g on M.

Our analysis is based on Kato's inequality [11] which we prove for the present situation, and the ensuing domination of the semigroup $\exp t\, D_\nabla$ by $\exp t\Delta_g$, [8] (see also [18], [19] for the special case of the complex line bundle over $\mathbf{R}^n$). This domination leads to the comparison of the spectra in the form

$$(1.1) \qquad \operatorname{Tr} \exp t\, D_\nabla \leqslant n \operatorname{Tr} \exp t\Delta_g \quad (t \geqslant 0),$$

where n is the rank of V. Estimate (1.1) of course yields inequalities for the corresponding Riemann zeta functions.

We extend this result by considering second order (linear) differential operators on V which differ from D_∇ by a zero order differential operator, i.e., a strict vector bundle endomorphism.

As an application we will consider Laplace-de Rham operators Δ and spinor Laplacians $\not{\nabla}^2$. They differ from the appropriate D_∇ by a strict vector bundle endomorphism involving the curvature of the connection employed. To cover the most general case, both exterior forms and spinors are allowed to have coefficients in an arbitrary hermitian vector bundle with connection over M. We then compute the differences $\Delta - D_\nabla$ and $\not{\nabla}^2 - D_\nabla$, obtaining the corresponding Weitzenböck formulas.

This note resulted from our study of the behavior of Yang-Mills potentials, which are the Christoffel symbols of connections (over $\mathbf{R}^n$). Over arbitrary

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