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THE FOUR-VERTEX THEOREM IN HYPERBOLIC SPACE

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Let e_i, $i = 1, 2, 3$, be the natural frame field on Minkowski 3-space and $'D$ be the connection such that $'D_V W = (VW^i)e_i$. Using the metric $\langle , \rangle$ of the 3-space which has one minus sign, the hyperbolic plane is represented by $\langle x, x \rangle = -1$. Thus, x is a unit normal of the latter surface and we see that $'D_V x = V$. Denoting by D the induced connection on the hyperbolic plane we have for its tangent vectors

$$'D_V W = D_V W + \langle V, W \rangle x \; . \tag{1}$$

On account of (1) we find $R(U, V)W = -\langle V, W \rangle U + \langle U, W \rangle V$, and hence the curvature of our surface is indeed -1.

If T and N designate the unit tangent and normal of a curve in the hyperbolic plane we know that $D_T T = \kappa N$ and $D_T N = -\kappa T$. Now because of (1) $'D_T T = \kappa N + x$. But $'D_T T = {'\kappa}'N$, where $'\kappa$ is the space curvature and $'N$ the space normal to the curve. We therefore record for later reference

$$('\kappa)^2 = \kappa^2 - 1 \; . \tag{2}$$

Also, if s stands for arc length we infer from (1) that

$$'D_T N = N'(s) = D_T N = -\kappa T = -\kappa x'(s) \; . \tag{3}$$

Through the two vertices which an oval necessarily has we draw a straight line whose equation is $\langle c, x \rangle = 0$. Then with all integrals taken around the oval we conclude in the usual manner with the aid of (3) that

$$\oint \langle c, x \rangle \kappa'(s) ds = -\oint \langle c, x'(s) \rangle \kappa \, ds = \oint \langle c, N'(s) \rangle ds = 0 \; .$$

This establishes the essence of the proof due to Herglotz [2, p. 201].

We now apply our methods to hyperbolic 3-space. In the imbedding Minkowski 4-space we see that $('\kappa)^2 = \langle T'(s), T'(s) \rangle$ is equivalent to

$$('\kappa)^2 = (\langle x', x' \rangle \langle x'', x'' \rangle - \langle x', x'' \rangle^2) / \langle x', x' \rangle^3 \; , \tag{4}$$

where the primes indicate differentiation with respect to some parameter u.

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