J. DIFFERENTIAL GEOMETRY
52 (1999) 453-609

GR = SW : COUNTING CURVES AND CONNECTIONS

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The purpose of this article is to present a proof of the assertion that a compact, symplectic 4-manifold has its Seiberg-Witten invariants equal to its Gromov invariants. In this regard, remark that the original Seiberg-Witten invariants are defined for any smooth, compact, oriented 4-manifold; and they are determined by the underlying differentiable structure when the Betti number b^{2+} is larger than 1. After the choice of orientation for the real line $\det^+ = H^0 \otimes \det(H^1) \otimes \det(H^{2+})$, the Seiberg-Witten invariants constitute a map from the set, $\mathcal{S}$, of $\mathrm{Spin}^{\mathbb{C}}$ structures on the 4-manifold to the integers. There is also an extension of SW in the case where the Betti number b^1 is positive to a map SW: $\mathcal{S} \to \Lambda^* H^1(X;\mathbb{Z})$. Here,

$$\Lambda^* H^1(X;\mathbb{Z}) = \mathbb{Z} \oplus H^1 \oplus \Lambda^2 H^1 \oplus \cdots \oplus \Lambda^{b_1} H^1.$$

Note that the projection of the image of SW on the summand $\mathbb{Z}$ reproduces the original map as defined from $\mathcal{S}$ to $\mathbb{Z}$. In either guise, this map, SW, is computed by an algebraic count of solutions to a certain non-linear system of differential equations on the manifold.

As remarked in [25], a symplectic manifold has a natural orientation as does the line $\det^+$. Furthermore, there is a canonical identification of the set $\mathcal{S}$ with $H^2(X;\mathbb{Z})$. Thus, on a symplectic 4-manifold, SW can be viewed as a map from $H^2(X;\mathbb{Z})$ to $\mathbb{Z}$, or, more generally, from $H^2(X;\mathbb{Z})$ to $\Lambda^* H^1(X;\mathbb{Z})$.

Meanwhile, a compact symplectic 4-manifold has a second natural map from $H^2(X;\mathbb{Z})$ to $\mathbb{Z}$, its Gromov invariant, Gr. The map Gr also

Received April 7, 1998. Supported in part by the National Science Foundation.