TOKYO J. MATH.
VOL. 17, NO. 1, 1994

Generating Function for the Spherical Functions on a Gelfand Pair of Exceptional Type

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(Communicated by T. Nagano)

Let X be a non-compact Riemannian symmetric space of rank 1. Then it is known that $X=G/K$, where G is a connected simple Lie group with finite center, K is a maximal compact subgroup of G, and if $G=KAN$ is an Iwasawa decomposition, we have $\dim A=1$. From the classification theory, it is known that X is either one of the classical hyperbolic spaces corresponding to the groups $SO_0(1,n)$, $SU(1,n)$ and $Sp(1,n)$ over the fields $\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{H}$ or the exceptional space corresponding to the exceptional simple group $F_{4(-20)}$, the so-called Cayley hyperbolic plane. Let M be the centralizer of A in K, then the Martin boundary K/M of $X=G/K$ is not a symmetric space, except for the case of real numbers. But, as is well known, (K, M) is a Gelfand pair, i.e. the convolution algebra of functions on K bi-invariant by M is commutative. A theory of the corresponding spherical functions is given in an exposé of Takahashi [5] for the classical case, while the exceptional case is treated in [6].

In the case of real hyperbolic spaces, the space K/M is the usual unit sphere S^{n-1} in $\boldsymbol{R}^n$, and we have the classical theory of spherical harmonics; the zonal spherical functions are given essentially by the Gegenbauer polynomials $C_p^{(n-2)/2}$ and we have the classical generating function expansion:

$$(1-2tx+t^2)^{-(n-2)/2}=\sum_{p=0}^{\infty} C_p^{(n-2)/2}(x)t^p\,, \qquad -1\leq x\leq 1\,, \quad -1<t<1\,,$$

which can be considered also as giving a generating function for the zonal spherical functions of the space $SO(n)/SO(n-1)$. In the papers [7], [8], we have shown that similar constructions are possible also in the other classical cases. The purpose of the present paper is to give a generating function in the exceptional case.

In what follows, we will follow the notations of [6].

The compact group K acts transitively on $\{F_2^u+F_3^v\,;\, u, v\in \boldsymbol{O}, |u|^2+|v|^2=1\}\cong S^{15}$ and the isotropy group of F_2^1 is the subgroup $M\subset L$, that is $K/M\cong S^{15}$, and it's identification is given by $kM\mapsto kF_2^1=F_2^u+F_3^v$. See §4 iv) in [6].

Received December 22, 1992