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Complex Multiplication Formulae for Hyperelliptic Curves of Genus Three

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Introduction.

Let $\wp(u)$ be a Weierstrass elliptic function satisfying $\wp'(u)^2=4\wp(u)^3-1$. Let $\zeta:=e^{2\pi i/3}$. Then $\wp(u)$ has a property $\wp(-\zeta u)=\zeta\wp(u)$. If b is an element of $\mathbf{Z}[\zeta]$, the integer ring generated by ζ, we have a b-multiplication formula of $\wp(u)$. If b is a prime element and $b\equiv 1 \bmod 3$, the b-multiplication formula is of the form

$$\wp(bu)=\frac{\wp(u)(\wp(u)^{\mathrm{N}b-1}+\cdots+b)}{(b\wp(u)^{(\mathrm{N}b-1)/2}+\cdots\pm 1)^2}, \tag{0.1}$$

and all the coefficients belong to $\mathbf{Z}[\zeta]$. (These facts seem to be already known to Eisenstein [6]). Therefore the product of the roots $\{\wp(u)\}$ except for 0 of the numerator is equal to $\pm b$, and the product of reciprocals of the roots $\{\wp(u)\}$ of the denominator is equal to b^2. So we have factorization of b or b^2 in an extended integer ring of $\mathbf{Z}[\zeta]$. Analogous fact is known for a function $\wp(u)$ satisfying $\wp'(u)^2=4\wp(u)^3-\wp(u)$.

By using these facts essentially, the cubic and quartic Gauss sums were deeply investigated (see [12] and [13]). So it seems natural for us to expect the existence of formulae analogous to (0.1) for curves of higher genus. A remarkable formula was discovered by D. Grant for the curve of genus two defined by $y^2=x^5+1/4$ ([9]).

The purpose of this paper is to generalize his formula. Let C be a curve of genus g (≥ 1) defined by $y^2=f(x)$, where $f(x)$ is a polynomial of degree $2g+1$. Let J denote the Jacobian variety of the curve C, and $\iota: C \hookrightarrow J$ the canonical embedding. We identify J with a complex torus $\mathbf{C}^g/\Lambda$ where Λ is a lattice of $\mathbf{C}^g$. Let $u=(u_1, \cdots, u_g)$ be the canonical coordinate system of $\mathbf{C}^g$, and $\varphi(u)$ a meromorphic function on $\mathbf{C}^g/\Lambda$. We assume that $\varphi(u)$ satisfies $\varphi(-u)=-\varphi(u)$, because the Abelian functions $\varphi(u)$ we treat in this paper are odd or even functions. In the below, we denote by $x(u)$ and $y(u)$ the values of x-coordinate and y-coordinate, respectively, at u such that $u\in\iota(C)$. Then the restriction to $\iota(C)$ of the map $u\mapsto\varphi(bu)$ gives an algebraic function. Hence $\varphi\circ\iota$ has a

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