TOKYO J. MATH.
VOL. 23, NO. 2, 2000

On Sectional Curvature of Boggino-Damek-Ricci Type Spaces

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(Communicated by T. Nagano)

1. Introduction.

Boggino [B] proved that simply connected solvable Lie groups associated to 1 dimensional extensions of Lie algebras of Heisenberg type admit Einstein metrics with non-positive sectional curvature. These spaces contain the class of non-compact symmetric space of rank 1 and are called Damek-Ricci spaces after Damek and Ricci proved that these are harmonic spaces.

We consider a class of solvable Lie groups which includes Damek-Ricci spaces. Let $\{\mathfrak{n}, \langle\ ,\ \rangle_{\mathfrak{n}}\}$ be a 2-step nilpotent metric Lie algebra, $\mathfrak{a}$ a 1 dimensional real vector space and A a non-zero vector in $\mathfrak{a}$. We denote the center of $\mathfrak{n}$ and the orthogonal complement of the center in $\mathfrak{n}$ by $\mathfrak{z}$ and $\mathfrak{v}$ respectively. For $k \in \mathbf{R}^+$ we define a representation f of $\mathfrak{a}$ on $\mathfrak{n}$ by

$$f(A)V = \frac{k}{2}V \quad f(A)Z = kZ \quad \text{for all } V \in \mathfrak{v} \text{ and } Z \in \mathfrak{z}.$$

Since $\mathfrak{a}$ acts on $\mathfrak{n}$ as a derivation by f, the semi-direct sum $\mathfrak{s}_k(A; \mathfrak{n}) = \mathfrak{n} \times_f \mathfrak{a}$ of $\mathfrak{n}$ and $\mathfrak{a}$ becomes a solvable Lie algebra. We define an inner product $\langle\ ,\ \rangle_{\mathfrak{a}}$ on $\mathfrak{a}$ by $\langle A, A\rangle_{\mathfrak{a}} = 1$ and an inner product $\langle\ ,\ \rangle$ on $\mathfrak{s}_k(A; \mathfrak{n})$ by the direct sum of $\langle\ ,\ \rangle_{\mathfrak{a}}$ and $\langle\ ,\ \rangle_{\mathfrak{n}}$. We consider the simply connected Lie group with the induced left invariant metric asociated to $\{\mathfrak{s}_k(A; \mathfrak{n}), \langle\ ,\ \rangle\}$. We denote it by $\{S_k(A; \mathfrak{n}), g\}$ and call it a Boggino-Damek-Ricci type space (abbreviated to a BDR-type space).

Mori [M] and Yamada [Y] studied existence of Einstein metrics with non-positive sectional curvature in BDR-type spaces. By the result of Heintze [H], BDR-type spaces have non-positive sectional curvature for sufficient large k, and we can ask the following question:

Can we determine the smallest value of k such that sectional curvature of BDR-type space is non-positive?

In this paper we answer this question in the case that the nilpotent part of BDR-type space is a Lie algebra of echelon type.

Received March 18, 1999
Revised November 15, 1999