TOKYO J. MATH.
VOL. 24, NO. 1, 2001

Hopf-type Estimates for Viscosity Solutions to Concave-Convex Hamilton-Jacobi Equations

NGUYEN DUY Thai Son

Kyoto Sangyo University

(Communicated by H. Ishii)

1. Introduction.

Consider the Cauchy problem for the simplest Hamilton-Jacobi equation, namely,

$$\partial u/\partial t + f(\partial u/\partial x) = 0 \quad \text{in} \quad \mathcal{D} := \{t > 0, x \in \mathbf{R}^n\}, \tag{1}$$

$$u(0, x) = \phi(x) \quad \text{on} \quad \{t = 0, x \in \mathbf{R}^n\}. \tag{2}$$

Here, $\partial/\partial x := (\partial/\partial x_1, \cdots, \partial/\partial x_n)$. Let $\mathrm{Lip}(\overline{\mathcal{D}}) := \mathrm{Lip}(\mathcal{D}) \cap C(\overline{\mathcal{D}})$, where $\mathrm{Lip}(\mathcal{D})$ is the set of all locally Lipschitz continuous functions $u = u(t, x)$ defined on $\mathcal{D}$. A function $u \in \mathrm{Lip}(\overline{\mathcal{D}})$ will be called a *global solution* of the Cauchy problem (1)–(2) if it satisfies (1) almost everywhere in $\mathcal{D}$, with $u(0, \cdot) = \phi$. A global solution of (1)–(2) is given by explicit formulas of Hopf [3] in the following two cases:

(a) f convex (or concave) and ϕ largely arbitrary; and

(b) ϕ convex (or concave) and f largely arbitrary.

It is unlikely that such restrictions, either on the Hamiltonian $f = f(p)$ or on the initial data $\phi = \phi(x)$, are really vital. A relevant solution is expected to exist under much wider assumptions. According to Hopf, that he has been unable to get further is doubtless due to a limitation in his approach: he uses the Legendre transformation globally, and this global theory has been carried through only in the case of convex (or concave) functions (Fenchel's theory of conjugate convex (or concave) functions).

We shall often suppose in this note that $n = n_1 + n_2$ and that the variable $p \in \mathbf{R}^n$ is separated into two as $p = (p', p'')$ with $p' \in \mathbf{R}^{n_1}$, $p'' \in \mathbf{R}^{n_2}$. (Similarly for $x, z, \cdots \in \mathbf{R}^n$. In particular, the zero-vector in $\mathbf{R}^n$ will be $0 = (0', 0'')$, where $0'$ and $0''$ stand for the zero-vectors in $\mathbf{R}^{n_1}$ and $\mathbf{R}^{n_2}$, respectively.) Recall (see Rockafellar [6]) that a function $f = f(p', p'')$ is called *concave-convex* if it is concave in $p' \in \mathbf{R}^{n_1}$ for each $p'' \in \mathbf{R}^{n_2}$ and convex in $p'' \in \mathbf{R}^{n_2}$ for each $p' \in \mathbf{R}^{n_1}$. We have proposed in [7] to examine a class of

Received February 14, 2000
Revised June 27, 2000
On leave from Hue University of Vietnam. Supported by the JSPS Postdoctoral Fellowship for Foreign Researchers in Japan.