On a unit group generated by special values of Siegel modular functions. II

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1. Introduction. In the preceding paper [1], we constructed a group of units with full rank for the ray class field k_6 of $\mathbf{Q}(\exp(2\pi i/5))$ modulo 6 using special values of Siegel modular functions and circular units. Our work was based on Shimura's reciprocity law [3] which describes explicitly the Galois action on the special values of theta functions and numerical computation. In this paper, we construct certain units of the ray class field k_{18} of $\mathbf{Q}(\exp(2\pi i/5))$ modulo 18.

2. Siegel modular functions. We argue in a situation similar to [1]. So we explain notations briefly. We denote as usual by $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ by the ring of rational integers, the field of rational numbers, real numbers and complex numbers, respectively. For a positive integer n, let I_n be the unit matrix of dimension n and $\zeta_n = \exp(2\pi i/n)$. Let $\mathfrak{S}_2$ be the set of all complex symmetric matrices of degree 2 with positive definite imaginary parts. For $u \in \mathbf{C}^2$, $z \in \mathfrak{S}_2$ and $r, s \in \mathbf{R}^2$, put as usual

$$\Theta(u, z; r, s) = \sum_{x \in \mathbf{Z}^2} e\left(\frac{1}{2}{}^t(x+r)z(x+r) + {}^t(x+r)(u+s)\right),$$

where $e(\xi) = \exp(2\pi i \xi)$ for $\xi \in \mathbf{C}$. Let N be a positive integer. If we define

$$\Phi(z; r, s; r_1, s_1) = \frac{2\Theta(0, z; r, s)}{\Theta(0, z; r_1, s_1)}$$

for $r, s, r_1, s_1 \in (1/N)\mathbf{Z}^2$, then $\Phi(z; r, s; r_1, s_1)$ is a Siegel modular function of level $2N^2$.

Let $\Gamma_1 = S_p(2, \mathbf{Z}) = \{\alpha \in GL_4(\mathbf{Z}) \mid {}^t\alpha J \alpha = J\}$, where

$$J = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}.$$

We let every element

$$\alpha = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

act on $\mathfrak{S}_2$ by $\alpha(z) = (Az+B)(Cz+D)^{-1}$ for $z \in \mathfrak{S}_2$.

If α is a matrix in $M_4(\mathbf{Z})$ such that ${}^t\alpha J \alpha = vJ$ and $\det(\alpha) = v^2$ with positive integer v prime to $2N^2$, then there exists a matrix β_α in Γ_1 with

$$\alpha \equiv \begin{pmatrix} I_2 & 0 \\ 0 & vI_2 \end{pmatrix} \beta_\alpha \pmod{2N^2}.$$

We let α act on $\Phi(z; r, s; r_1, s_1)$ by $\Phi^\alpha(z; r, s; r_1, s_1) = \Phi(\beta_\alpha(z); r, vs; r_1, vs_1)$. Then Φ^α is also a Siegel modular function of level $2N^2$.

In what follows, we fix $\zeta = \zeta_5$ and $k = \mathbf{Q}(\zeta)$. Let σ be the element of the Galois group $G(k/\mathbf{Q})$ such that $\zeta^\sigma = \zeta^2$ and define the endomorphism φ of $k^\times$ by $\varphi(a) = a^{1+\sigma^3}$ for $a \in k^\times$. Furthermore put

$$\begin{aligned} z_0 &= \begin{pmatrix} \zeta^2+\zeta^4 & \zeta^3 \\ \zeta^4+\zeta^3 & \zeta \end{pmatrix}^{-1} \begin{pmatrix} -\zeta & \zeta^4 \\ -\zeta^2 & \zeta^3 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 2+\zeta-\zeta^3-2\zeta^4 & 2-\zeta+\zeta^2-2\zeta^3 \\ 2-\zeta+\zeta^2-2\zeta^3 & \zeta+2\zeta^2-2\zeta^3-\zeta^4 \end{pmatrix}. \end{aligned}$$

We note that z_0 is a CM-point associated to a Fermat curve $y^2 = 1 - x^5$. For an element ω in the integer ring $\mathfrak{O}_k$ of k, let $R(\omega) \in M_4(\mathbf{Z})$ be the regular representation of ω with respect to the basis $\{-\zeta, \zeta^4, \zeta^2+\zeta^4, \zeta^3\}$. Then, $R(\varphi(\omega))z_0 = z_0$, ${}^tR(\varphi(\omega))JR(\varphi(\omega)) = vJ$ and $\det R(\varphi(\omega)) = v^2$, where $v = N_{k/\mathbf{Q}}(\omega)$.

3. Structure of the Galois group. For a positive integer N, we denote by k_N the ray class field of k modulo N. We explain the structure of the Galois group $G(k_{18}/k)$ which is needed for our argument. For a positive integer m, we put $S_m = \{a \in k^\times | a \equiv 1 \pmod{m}\}$ and $\tilde{S}_m = \{(a) | a \in S_m\}$, where (a) is the principal ideal of k generated by a.

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