Existence of bounded solutions for semilinear degenerate elliptic equations with absorption term

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1. Introduction. Let $N \geq 1$ and $p > 1$. Let F be a compact set and Ω be a bounded open set of $\boldsymbol{R}^N$ satisfying $F \subset \Omega \subset \boldsymbol{R}^N$. We also set $\Omega' = \Omega \setminus \partial F$, where $\partial F = F \setminus \mathring{F}$ and $\mathring{F}$ denotes the interior of F. Define

$$(1.1) \qquad P = -\operatorname{div}(A(x)\nabla\cdot),$$

where $A(x) \in C^1(\Omega')$ is positive in $\Omega \setminus F$ and vanishes in $\mathring{F}$. First we shall consider removable singularities of solutions for degenerate semilinear elliptic equations. Assume that $u \in C^0(\Omega') \cap C^2(\Omega \setminus F)$ satisfies

$$(1.2) \quad Pu + B(x)Q(u) = f(x), \qquad \text{in } \Omega',$$

for $f/B \in L^\infty(\Omega)$. Here $Q(u)$ is a nonlinear term defined in the section 2. Then we shall show the existence of a bounded solution in Ω which coincides with u in $\Omega' = \Omega \setminus \partial F$. This result was established by H. Brezis and L. Veron in [2], under the assumptions that F consists of finite points, $Q(t) = |t|^{p-1}t$ and $A(x)$, $B(x)$, $C(x)$ are positive constants. (see also [5]). In this paper we generalize their results for an arbitrary compact set F in place of finite set and for a wider class of (degenerate) elliptic operators P.

Secondly as an application, we shall consider the Dirichlet boundary problem for genuinely degenerate semilinear elliptic operators:

$$(1.3) \quad \begin{cases} Pu + B(x)Q(u) = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Then we shall establish the existence and uniqueness of bounded solutions u for this problem. When P is uniformly elliptic on $\bar{\Omega}$, this problem has been treated by many authors. But the development of the theory seems to be rather limitted in the study of genuinely degenerate operators.

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2. Main results and applications. Let $N \geq 1$. Let F and Ω be a compact set and bounded open subset of $\boldsymbol{R}^N$ respectively, satisfying $F \subset \Omega$, and set $\Omega' = \Omega \setminus \partial F$. Here ∂F is defined as $\partial F = F \setminus \mathring{F}$. In the next we define a modified distance to ∂F.

Definition 1. Let $d(x) \in C^\infty(\Omega')$ be a nonnegative function satisfying

$$(2.1) \quad C^{-1} \leq \frac{d(x)}{dist(x, \partial F)} \leq C, \qquad |\partial^\gamma d(x)| \leq C(\gamma) dist(x, \partial F)^{1-|\gamma|}, \; x \in \Omega',$$

where C and $C(\gamma)$ are positive numbers independent of each point x.

We suppose the following four assumptions:

[H-1] (*Coefficients*).

$$(2.2) \quad \begin{cases} A(x) \in C^1(\Omega') \cap L^1_{\mathrm{loc}}(\Omega), \\ A(x) = 0 \text{ in } \mathring{F} = F \setminus \partial F, \\ A(x) > 0 \text{ in } \Omega \setminus F, \end{cases} \qquad \begin{cases} B(x) \in L^\infty_{\mathrm{loc}}(\Omega') \cap L^1_{\mathrm{loc}}(\Omega), \\ B(x) > 0 \text{ in } \Omega' = \Omega \setminus \partial F, \\ C(x) \in L^\infty_{\mathrm{loc}}(\Omega') \cap L'_{\mathrm{loc}}(\Omega), \\ C(x) \geq 0 \text{ in } \Omega. \end{cases}$$

[H-2] (*Nonlinear term*).

$$(2.3) \quad \begin{cases} Q(t) \text{ is monotone increasing and continuous on } \mathbf{R} \\ \text{such that } Q(0) = 0 \text{ and } Q(t)t > 0 \text{ for any } t \in \mathbf{R} \setminus \{0\}. \end{cases}$$

Definition 2. Let us set for $x \in \Omega' = \Omega \setminus \partial F$

$$(2.4) \quad \tilde{A}(x) = A(x) + d(x)|\nabla A(x)|, \qquad \Phi(x) = \text{ess-sup}_{|y-x|<\frac{d(x)}{2}} \frac{\tilde{A}(y)}{B(y)}.$$

[H-3]. There is a positive number $\delta_0 > 0$ such that

$$(2.5) \quad \sup_{t \in \mathrm{R} \setminus \{0\}} \frac{|t|^{1+\delta_0}}{|Q(t)|} < +\infty, \text{ Super-linearlity.}$$

and

$$(2.6) \quad \liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{0<d(x)<\varepsilon} \tilde{A}(x) \left[\left(\frac{\Phi(x)}{d(x)^2} \right)^{\frac{1}{\delta_0}} + 1 \right] \frac{dx}{d(x)} < +\infty.$$