A remark on 2-microhyperbolicity

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Introduction. The notion of microhyperbolicity due to Kashiwara-Kawai [3] has played an important role in many problems concerning existence and regularity of solutions of linear PDE in microfunctions. When we study microfunction solutions from a second microlocal point of view, new phenomena appear. In fact, the basic notion is that of the second wave front set, and this notion is already for opportunity reasons closely related to that of second-hyperfunctions. respectively second-microfunctions. (We shall call them 2-hyperfunctions and 2-microfunctions henceforth). It is then natural to introduce the notion of 2-microhyperbolicity, a notion which turns out to be quite efficient for regularity problems (cf. N. Tose [11]) but which seems to have no immediate applications for existence problems in standard microfunctions, in view of the fact that the space of 2-hyperfunctions is much larger than is the space of microfunctions.

The main purpose of the present paper is to clarify the situation by examining an example which is modelled on the classical Mizohata operator. Indeed, this operator shall be 2-microhyperbolic (with respect to some regular involutive homogeneous submanifold in the phase space), it will be solvable in 2-hyperfunctions, but we shall see that it is not solvable in standard microfunctions (see Theorem 2.1 below). Since we think that the primary object of study should always be the "equation" and since we "believe" in the importance of the notion of 2-microhyperbolicity, we expect that this example gives an additional argument in favour of 2-hyperfunctions as the correct frame in which one should perform 2-microlocal arguments.

1. Preliminaries. 1.1. 2-microlocal analysis. Since the problem is microlocal, we take, from the beginning, a local model V of a regular involutive homogeneous submanifold in the cotangent bundle $\sqrt{-1}T^*\mathbf{R}^n$ defined by

(1) $V := \{(x;\sqrt{-1}\xi\cdot dx) \in \sqrt{-1}T^*\mathbf{R}^n;\ \xi_1 = \cdots = \xi_d = 0\}.$

Here $x = (x_1, \ldots, x_n)$ is a system of coordinates in $\mathbf{R}^n$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n$ are the dual coordinates. We take the complexification of V in $T^*\mathbf{C}^n$ defined by

(2) $V^{\mathbf{C}} := \{(z;\zeta\cdot dz) \in T^*\mathbf{C}^n;\ \zeta_1 = \cdots = \zeta_d = 0\}.$

where $z = (z_1, \ldots, z_n)$ is a complex coordinate system of $\mathbf{C}^n$ corresponding to x and $\zeta = (\zeta_1, \ldots, \zeta_n)$ are the dual coordinates. Then a partial complexification $\tilde{V}$ of V in $V^{\mathbf{C}}$ is given by

(3) $\tilde{V} := \{(z;\zeta\cdot dz) \in T^*\mathbf{C}^n;\ \zeta_1 = \cdots = \zeta_d = 0,\ \Im z_{d+1} = \cdots = \Im z_n = 0,\ \Re\zeta_{d+1} = \cdots = \Re\zeta_n = 0\}.$

This space can be identified with the conormal bundle $T_N^*\mathbf{C}^n$ of

(4) $N := \{z \in \mathbf{C}^n;\ \Im z_{d+1} = \cdots = \Im z_n = 0\}.$

The space $\tilde{V}$ is endowed with the sheaf $\mathscr{C}_{\tilde{V}}$ of microfunctions with holomorphic parameters $z' = (z_1, \ldots, z_d)$ which is defined by

(5) $\mathscr{C}_{\tilde{V}} = H^{n-d}(\mu_N(\mathscr{O}_{\mathbf{C}^n}))$

by means of the Sato's microlocalization functor along N (refer to Kashiwara-Schapira [4] for this). First remark that the sheaf $\mathscr{A}_V^2 := \mathscr{C}_{\tilde{V}}|_V$ is a subsheaf of the sheaf $\mathscr{C}_{\mathbf{R}^n}$ of microfunctions on $\mathbf{R}^n$. Thus we have an exact sequence

(6) $0 \to \mathscr{A}_V^2 \to \mathscr{C}_{\mathbf{R}^n}|_V$

on V. To analyze the gap between the two sheaves, M. Kashiwara introduced the sheaf $\mathscr{C}_V^2$ of 2-microfunctions along V on $T_V^*\tilde{V}$ by

(7) $\mathscr{C}_V^2 := H^d(\mu_V(\mathscr{C}_{\tilde{V}})).$

The sheaf $\mathscr{C}_V^2$ gives rise to the exact sequences

(8) $0 \to \mathscr{C}_{\mathbf{R}^n}|_V \to \mathscr{B}_V^2,$

and

(9) $0 \to \mathscr{A}_V^2 \to \mathscr{B}_V^2 \to \mathring{\pi}_{V*}(\mathscr{C}_V^2) \to 0.$

Here the sheaf $\mathscr{C}_V^2$ restricted to the zero-section V of $T_V^*\tilde{V}$

(10) $\mathscr{B}_V^2 := \mathscr{C}_V^2|_V$

is the sheaf of 2-hyperfunctions, and

$$\mathring{\pi}_V : \mathring{T}_V^*\tilde{V} := T_V^*\tilde{V} \setminus V \to V$$

is the natural projection. It should be noted that the morphism $\mathscr{C}_{\mathbf{R}^n}|_V \to \mathscr{B}_V^2$ is not surjective, and we refer to Kataoka-Okada-Tose [5] for an expli-

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