# On the Diophantine Equation $2^aX^4 + 2^bY^4 = 2^cZ^4$

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**1. Introduction.** In this paper, an integer means a rational integer. The greatest common divisor of the integers $a$ and $b$ is denoted by $(a, b)$. We shall prove the following main theorems.

**Theorem 1.** *Let $a, b, c$ be non-negative integers. If $X, Y, Z$ is a solution of the equation*

$$2^aX^4 + 2^bY^4 = 2^cZ^4$$

*in positive odd integers, then*

$$X = Y = Z \text{ and } a + 1 = b + 1 = c.$$

**Theorem 2.** *Let $m$ be a non-negative integer. Then the equation*

$$X^4 + 2^mY^2 = Z^4$$

*has no solutions in nonzero integers $X, Y, Z$.*

**2. Preliminaries.** We remind first the following three theorems which are all well-known (see [1], [2] or [3]).

**Theorem 3.** *Let $X, Y, Z$ be a solution of the equation*

$$X^2 + Y^2 = Z^2$$

*with positive integers $X, Y, Z$ such that $(X, Y) = 1$ and $X$ odd. Then there exist unique positive integers $u$ and $v$ of opposite parity with $(u, v) = 1$ and $u > v > 0$ such that*

$$X = u^2 - v^2,$$
$$Y = 2uv,$$
$$Z = u^2 + v^2.$$

**Theorem 4.** *The equation*

$$X^4 + Y^4 = Z^2$$

*has no solutions in nonzero integers $X, Y, Z$.*

**Theorem 5.** *The equation*

$$X^4 + Y^2 = Z^4$$

*has no solutions in nonzero integers $X, Y, Z$.*

**3. On the equation $X^4 + 2^mY^4 = Z^4$**

In this section, we shall give a simple proof of the following theorem which is slightly stronger than, and implies Fermat's last theorem for $n = 4$ (see [4]).

**Theorem 6.** *Let $m$ be a non-negative integer. Then the equation*

$$X^4 + 2^mY^4 = Z^4$$

*has no solutions in odd integers $X, Y, Z$.*

*Proof.* Suppose that $u$ is the least integer for which

$$x^4 + 2^my^4 = u^4$$

has a solution in positive odd integers $x, y, u$ for some non-negative integer $m$. The statement that $u$ is least immediately implies that three integers $x, y, u$ are pairwise relatively prime. Since the fourth power of an odd integer is congruent to 1 modulo 16, we have

$$2^my^4 = u^4 - x^4 \equiv 1 - 1 = 0 \pmod{16}.$$

Then $m > 3$. Since $u$ and $x$ are both odd and relatively prime, we have

$$u^2 + x^2 \equiv 2 \pmod 4$$

and

$$(u^2 + x^2, u + x) = (u^2 + x^2, u - x)$$
$$= (u + x, u - x) = 2.$$

And since

$$2^my^4 = u^4 - x^4 = (u - x)(u + x)(u^2 + x^2),$$

there exist positive odd integers $a, b, c$ such that

$$u - x = 2a^4,\ u + x = 2^{m-2}b^4,\ u^2 + x^2 = 2c^4$$

or

$$u - x = 2^{m-2}b^4,\ u + x = 2a^4,\ u^2 + x^2 = 2c^4.$$

Hence

$$4c^4 = 2(u^2 + x^2) = (u - x)^2 + (u + x)^2$$
$$= 4a^8 + 2^{2m-4}b^8$$

and so we obtain

$$(a^2)^4 + 2^{2m-6}(b^2)^4 = c^4$$

in positive odd integers $a, b, c$.

Moreover, since $0 < x < u$, we have $c^4 < 2c^4 = u^2 + x^2 < 2u^2 < u^4$ and so $0 < c < u$. Thus $u$ was not least after all and the theorem is proved.

**4. Proofs of the main theorems.**

**Lemma 7.** *Let $X, Y, Z$ be a solution of the equation*

$$X^4 + Y^4 = 2Z^2$$

*in non-negative integers. Then*

$$X^2 = Y^2 = Z.$$

*Proof.* Let $X, Y, Z$ be a solution of the equation $X^4 + Y^4 = 2Z^2$ in non-negative integers. If one of $X$, $Y$ and $Z$ is zero, then $X = Y = Z = 0$. Thus, we suppose that $X$, $Y$ and $Z$ are