12. On Contiguity Relations of the Confluent Hypergeometric Systems

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Introduction. This paper concerns the contiguity relations for the confluent hypergeometric systems M_λ (CHG system, for short) defined on the space $Z_{r,n}$ of $r \times n$ complex matrices of maximum rank $r\,(< n)$. As for the definition of the CHG systems and notations employed in this paper, we adopt those of [7].

In [4], we gave a Lie algebra of contiguity operators (see Definition 2.1) in an explicit form. In the present paper, we show that the contiguity operators, obtained in [4], appear in a natural manner in connection with the root space decomposition of the Lie algebra $\mathfrak{gl}_n(\boldsymbol{C})$ with respect to the maximal abelian subalgebra $\mathfrak{h} = LieH_\lambda$.

1. Root space decomposition. Let $H = H_\lambda = J(\lambda_1) \times \cdots \times J(\lambda_l)$ be a maximal abelian subgroup of $GL(n, \boldsymbol{C})$ corresponding to the composition $\lambda = (\lambda_1, \ldots, \lambda_l)$ of n, where $J(\lambda_k)$ be the Jordan group of size λ_k.

In the following, we often decompose an $n \times n$ matrix X into blocks according to the composition λ as

$$X = (X_{ij})_{1 \le i,j \le l},$$

where X_{ij} is a $\lambda_i \times \lambda_j$ matrix, which will be called (i, j)-block of X.

We denote by $\mathfrak{h}$ the Lie algebra of H, which is given by

$$\mathfrak{h} = \left\{ h = \oplus_{i=1}^{l} h^{(i)}\ ;\ \ h^{(i)} = \sum_{k=0}^{\lambda_i - 1} h_k^{(i)} \Lambda_{\lambda_i}^k,\ h_k^{(i)} \in \boldsymbol{C} \right\}$$

and is a maximal abelian subalgebra of $\mathfrak{gl}_n = \mathfrak{gl}_n(\boldsymbol{C})$. The dual space of $\mathfrak{h}$ is denoted by $\mathfrak{h}^*$. For any $h \in \mathfrak{h}$, we consider an endmorphism $ad\ h : \mathfrak{gl}_n \to \mathfrak{gl}_n$ defined by

$$(ad\, h)X := [h, X] = hX - Xh.$$

We say that a non zero element $\beta \in \mathfrak{h}^*$ is a *root* for $\mathfrak{h}$ if the vector space

$$\mathfrak{g}_\beta := \{X \in \mathfrak{gl}_n\, ;\, (ad\, h - \beta(h))X = 0 \text{ for all } h \in \mathfrak{h}\}$$

is of dimension greater than or equal to 1. The vector space $\mathfrak{g}_\beta$ will be called the *root subspace*. Note that $\mathfrak{g}_0 = \mathfrak{h}$.

Let β_j $(j = 1, \ldots, l)$ be an element of $\mathfrak{h}^*$ which sends the matrix $\oplus_{k=1}^{l} (\sum_{i=0}^{\lambda_k - 1} h_i^{(k)} \Lambda_{\lambda_k}^i)$ to the common diagonal element $h_0^{(j)}$ of (j, j)-block. We see that the set Δ of non zero roots for $\mathfrak{h}$ is given by

$$\Delta = \{\beta_i - \beta_j\, ;\, i, j = 1, \ldots, l,\ i \neq j\}.$$

Proposition 1.1. *For any root* $\beta_i - \beta_j \in \Delta$,

$$\mathfrak{g}_{\beta_i - \beta_j} = \boldsymbol{C} X_{\beta_i - \beta_j},$$

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