11. *Higher Specht Polynomials for the Symmetric Group*

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(Communicated by Shokichi IYANAGA, M. J. A., Feb. 12, 1993)

§0. Introduction. We are concerning with constructing a basis of the S_n-module $H = \boldsymbol{Q}[x_1,\ldots,x_n]/(e_1,\ldots,e_n)$, where $(e_1,\ldots,e_n)$ denotes the ideal generated by elementary symmetric polynomials $e_j = e_j(x_1,\ldots,x_n)$ for $j = 1,\ldots,n$.

Let $P = \boldsymbol{Q}[x_1,\ldots,x_n]$ be the algebra of polynomials of n variables $x_1,\ldots,x_n$ with rational coefficients, on which the symmetric group S_n acts by the permutation of the variables:

$$(\sigma f)(x_1,\ldots,x_n) = f(x_{\sigma(1)},\ldots,x_{\sigma(n)}) \quad (\sigma \in S_n).$$

Let us denote by Λ the subalgebra of P consisting of the symmetric polynomials. Let $e_j(x_1,\ldots,x_n) = \sum_{1\le i_1<\cdots<i_j\le n} x_{i_1}\ldots x_{i_j}$ be the elementary symmetric polynomial of degree j and put $J_+ = (e_1,\ldots,e_n)$, an ideal generated by $e_1,\ldots,e_n$. The quotient algebra $H = P/J_+$ has a structure of an S_n-module. It is well known that the S_n-module H is isomorphic to the regular representation. In other words, every irreducible representation of S_n occurs in H with multiplicity equal to its dimension. We will give a combinatorial procedure to obtain a basis of each irreducible component of H.

For a Young diagram λ of n cells, one can construct an S_n-module $V(\lambda)$ as follows (cf. [5]). For a tableau T of shape λ put

$$\Delta_T = \prod_{\beta\ge 1} \Delta_T(\beta) \in P,$$

where $\Delta_T(\beta)$ is the product of differences $x_i - x_j$ for the pair $\{(i,j)\,;\, i<j\}$ appearing in the β-th column in T. The polynomial Δ_T is called the Specht polynomial of T. The space $V(\lambda)$ spanned by all the Specht polynomials Δ_T for tableaux T of shape λ is naturally equipped with a structure of an S_n-module. It is well known that $V(\lambda)$ is irreducible for any Young diagram λ and has a basis $\{\Delta_T\,;\, T \text{ is a standard tableau of shape } \lambda\}$.

Our basis of H is parametrized by the pair of standard tableaux (S, T) of the same shape and turns out to be a natural generalization of these standard Specht polynomials. One finds a related topic in [1].

§1. Standard tableaux and their indices. Fix a Young diagram $\lambda = (\lambda_1,\ldots,\lambda_n)\,(\lambda_1 \ge \cdots \ge \lambda_n \ge 0)$ consisting of n cells. We often say that λ is a partition of n and write $\lambda \vdash n$. The set of tableaux (resp. standard tableaux) of shape λ is denoted by $Tab(\lambda)$ (resp. $STab(\lambda)$) (cf. [5]). For a standard tableau S of shape λ, one can associate the index tableau $i(S)$ of the same shape in the following manner (cf. [2]). Define the word $w(S)$ by reading S from the bottom to the top in consecutive columns, starting from the left. The number 1 in the word $w(S)$ has index 0. If the number k in the word has index p, then $k+1$ has index p or $p+1$ according as it lies to