72. An Additive Problem of Prime Numbers. III

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(Communicated by Shokichi IYANAGA, M. J. A., Oct. 14, 1991)

§1. Let γ run over the imaginary parts of the zeros of the Riemann zeta function $\zeta(s)$. We assume the Riemann Hypothesis throughout this article. Here we are concerned with the value distribution of the bounded oscillating quantity $G(X)$ for $X\geq 1$ defined by

$$G(X)\equiv\Re\left\{\sum_{\gamma>0}\frac{X^{i\gamma}}{(1/2+i\gamma)(3/2+i\gamma)}\right\}.$$

This function plays important roles in some problems in the analytic theory of numbers. We may recall two formulas involving $G(X)$. One is concerned with Goldbach's problem on average and the other is concerned with the prime number theorem on average.

(I) *For $X>X_0$, we have*

$$\sum_{n\leq X}\left\{\sum_{m+k=n}\Lambda(m)\Lambda(k)-n\cdot\prod_{p|n}\left(1+\frac{1}{p-1}\right)\prod_{p\nmid n}\left(1-\frac{1}{(p-1)^2}\right)\right\}$$
$$=-4X^{3/2}G(X)+O((X\log X)^{1+1/3}),$$

where $\Lambda(n)$ is the von Mangoldt function.

(II) *For $X\geq 1$, we have*

$$\int_0^X(\sum_{n\leq y}\Lambda(n)-y)\,dy=-2X^{3/2}G(X)-X\log(2\pi)+\log(2\pi)+C_0$$
$$-1-(6/\pi^2)\zeta'(2)-X\sum_{a=1}^{\infty}(X^{-2a}/2a(2a-1)),$$

where C_0 is the Euler constant.

(I) has been proved in the author's previous work [7]. (II) is known to hold without assuming any unproved hypothesis in the following form (cf. p. 52 and p. 74 of Edwards [5]). For $X\geq 1$,

$$\int_0^X(\sum_{n\leq y}\Lambda(n)-y)\,dy=-\sum_{\substack{\zeta(\rho)=0\\0<\Re(\rho)<1}}\frac{X^{\rho+1}}{\rho(\rho+1)}-X\sum_{a=1}^{\infty}\frac{X^{-2a}}{2a(2a-1)}-\frac{\zeta'}{\zeta}(0)X+\frac{\zeta'}{\zeta}(-1).$$

In (II), $G(X)$ is the only oscillating part. However in (I), the remainder term has still another oscillating property connected with the distribution of the zeros of $\zeta(s)$ as has been seen in [6] and [7].

We notice that the formula (II) implies, for example, that

$$G(1)=(1/2)(-(1/2)+C_0-(6/\pi^2)\zeta'(2)-\log 2)$$

and

$$G(2)=(1/4\sqrt{2})(1-\log\pi+C_0-(6/\pi^2)\zeta'(2)+\log 2-(3/2)\log 3).$$

Generally, we have for $X>1$,

$$G(X)+(1/2X^{3/2})\{(X-1)\log\pi-C_0+(6/\pi^2)\zeta'(2)\}-(1/2X^{3/2})\{(X^2/2)-1\}$$
$$=-(1/2X^{3/2})\{(X-1)\log 2+\log A_1+(X-[X])\log A_2$$
$$-\log(1-(1/X))+((X+1)/2)\log(1-(1/X^2))\},$$