11. On Pathwise Projective Invariance of Brownian Motion. III

By Shigeo TAKENAKA*)
Department of Mathematics, Nagoya University

(Communicated by Kôsaku YOSIDA, M. J. A., Feb. 13, 1990)

In part I we demonstrated that the group $SL(2, \boldsymbol{R})$ acts on the path space of Brownian motion $\{B(t);\ t \in \boldsymbol{R}\}$ as

$$B^g(t;\omega)=(ct+d)B\left(\frac{at+b}{ct+d};\omega\right)-ctB\left(\frac{a}{c};\omega\right)-dB\left(\frac{b}{d};\omega\right),\qquad g=\begin{pmatrix} a, & b \\ c, & d \end{pmatrix} \in SL(2, \boldsymbol{R}), \tag{1}$$

and that the above action is compatible with the group action,

$$(B^g)^h(t;\omega)=B^{gh}(t;\omega).$$

In this part we obtain analogous invariance properties of multi-parameter Brownian motions.

§8. **Multi-parameter Brownian motion.** A real Gaussian system $\{B(\boldsymbol{t});\ \boldsymbol{t} \in \boldsymbol{R}^n\}$ is called a Brownian motion if it satisfies the following conditions ([3]):

($\mathscr{B}$1) $\qquad B(\boldsymbol{0})=0,$

($\mathscr{B}$2) $\qquad B(\boldsymbol{s})-B(\boldsymbol{t})\simeq N(0, |\boldsymbol{s}-\boldsymbol{t}|)$

and

($\mathscr{B}$3) $\qquad B(\boldsymbol{t})$ is continuous in $\boldsymbol{t}$.

It is easy to see that the following transformed processes $B_{1,v}$, $B_{2,u}$ and $B_{4,g}$ satisfy the conditions ($\mathscr{B}$1)–($\mathscr{B}$3). That is, all these processes are Brownian motions.

($\mathscr{T}$1) $\qquad B_{1,\boldsymbol{v}}(\boldsymbol{t})\equiv B(\boldsymbol{t}+\boldsymbol{v})-B(\boldsymbol{v}), \qquad \boldsymbol{v} \in \boldsymbol{R}^n$ (shift),

($\mathscr{T}$2) $\qquad B_{2,u}(\boldsymbol{t})\equiv e^{-u/2}B(e^u\boldsymbol{t}), \qquad u \in \boldsymbol{R}^1$ (homogeneous dilation)

and

($\mathscr{T}$4) $\qquad B_{4,g}(\boldsymbol{t})\equiv B(g\cdot\boldsymbol{t}), \qquad g \in SO(n)$, (rotation).

A difficulty occurs when we consider about multi-parameter analogue of the transform ($\mathscr{T}$3). Let us recall that in the case of 1-parameter Brownian motion, we employed the transformation

($\mathscr{T}$3) $\qquad B_3(t)=t\cdot B\left(-\frac{1}{t}\right), \qquad t \in \boldsymbol{R}^1$

as the projective inversion and obtained the pathwise projective invariance. We can consider this inversion map as a coordinate map between 0-neighborhood $\{(x, y) \in \boldsymbol{P}^1,\ y\neq 0\}$ and ∞-neighborhood $\{(x, y);\ x\neq 0\}$ of the 1-dimensional real projective space $\boldsymbol{P}^1$,

*) Current Address: Department of Mathematics, Hiroshima University.