84. On the Hasse Norm Principle for Certain Generalized Dihedral Extensions over Q

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(Communicated by Shokichi IYANAGA, M. J. A., Dec. 12, 1990)

Introduction. Let l be an odd prime number and put $l^* := (-1)^{(l-1)/2} l$. Set $k := Q(\sqrt{l^*})$ and let K be the Hilbert class field of k. In this note, we study the Hasse norm principle for the Galois extension K/Q whose Galois group is a generalized dihedral group. More precisely, we express the number knot group for K/Q in terms of the ideal class group of k. Our theorem says that the validity of the Hasse norm principle for K/Q is equivalent to that for K/k. As an application, we determine the Ono invariant $E(K/Q)$ ([3, 4]), which was the motivation of this work.

§ 1. The number knot group $Ш(K/Q)$. For a finite Galois extension L/F of number fields, we denote by $Ш(L/F)$ the number knot group $F^\times \cap NJ_L/NL^\times$, where J_L is the idele group of L and N means the norm map in the obvious sense. Clearly, $Ш(L/F)=\{0\}$ is equivalent to the fact that the Hasse norm principle holds for L/F and we also remark that $Ш(L/F)$ is nothing but the Tate-Shafarevich group of the norm torus $T := \operatorname{Ker}(R_{L/F}(G_m) \xrightarrow{N} G_m)$.

First, let us recall Tate's cohomological method ([6]) to study $Ш(L/F)$ for a finite Galois extension L/F of number fields with the Galois group $G := Gal(L/F)$. (See, for example [5].)

By the exact sequence of G-modules

$$0 \longrightarrow L^\times \longrightarrow J_L \longrightarrow C_L \longrightarrow 0 \tag{1.1}$$

where $C_L := J_L/L^\times$, we have an exact sequence of Tate cohomology groups

$$\cdots \longrightarrow \hat{H}^{-1}(G, J_L) \xrightarrow{f} \hat{H}^{-1}(G, C_L) \longrightarrow \hat{H}^0(G, L^\times) \xrightarrow{g} \hat{H}^0(G, J_L) \longrightarrow \cdots. \tag{1.2}$$

Here it is easy to see that

$$\operatorname{Coker} f \simeq \operatorname{Ker} g = Ш(L/F). \tag{1.3}$$

If we choose a place w of L lying over each place v of F and denote by G_w the decomposition group of w, then we have the following commutative diagram:

$$\begin{tikzcd}
\prod_v H_2(G_w, Z) \arrow[r, "\phi"] \arrow[d, "\wr"'] & H_2(G, Z) \arrow[d, "\wr"'] \\
\prod_v \hat{H}^{-3}(G_w, Z) \arrow[r, "\psi"] \arrow[d, "\wr"', "\prod_v \cup \alpha_w"] & \hat{H}^{-3}(G, Z) \arrow[d, "\wr"', "\cup \alpha"] \\
\hat{H}^{-1}(G, J_L) = \prod_v \hat{H}^{-1}(G_w, L_w^\times) \arrow[r, "f"] & \hat{H}^{-1}(G, C_L)
\end{tikzcd} \tag{1.4}$$

where ϕ and ψ are the sum of the corestrictions with respect to $G_w \xrightarrow{\subset} G$