72. On the Divisor Function and Class Numbers of Real Quadratic Fields. II

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(Communicated by Shokichi IYANAGA, M. J. A., Nov. 9, 1990)

Abstract: The purpose of this paper is to continue work begun in [12] by providing lower bounds for the class numbers of real quadratic fields $\boldsymbol{Q}(\sqrt{d})$ in terms of the divisor function. These results generalize those of Halter-Koch in [5] as well as Azuhata [1]-[2], Mollin [7]-[11], and Yokoi [17]-[23].

§1. **Notation and preliminaries.** Throughout d is a positive square-free integer, and $K=\boldsymbol{Q}(\sqrt{d})$, and $h(d)$ is the class number of K. The maximal order in K is denoted $\mathcal{O}_K$, and the discriminant of K is $\Delta=4d/\sigma^2$ where $\sigma=\begin{Bmatrix}2 \text{ if } d\equiv 1 \pmod 4 \\ 1 \text{ if } d\equiv 2, 3 \pmod 4\end{Bmatrix}$. Let $w_d=(\sigma-1+\sqrt{d})/\sigma$.

If $[\alpha, \beta]$ is the module $\{\alpha x+\beta y : x, y \in Z\}$ then we observe that the maximal order $\mathcal{O}_K=[1, w_d]$. It can be shown (for example see Ince [6, pp. v–vii]) that I is an ideal in $\mathcal{O}_K$ if and only if $I=[a, b+cw_d]$ where a, b, $c \in Z$ (the rational integers) with $c|b$, $c|a$ and $ac|N(b+cw_d)$; where N is the norm from K to $\boldsymbol{Q}$. Moreover if $a>0$ then a is unique and is the smallest positive rational integer in I, denoted $a=L(I)$. Thus $N(I)=cL(I)$. If $c=1$ we say that I is a primitive *ideal*, and so $N(I)=L(I)$. Since $I=(c)[a/c, b/c+w_d]$ then we may restrict our attention to primitive ideals, (where (c) denotes the principal ideal generated by (c)).

A primitive ideal I is called *reduced* if it does *not* contain any non-zero element α such that both $|\alpha|<N(I)$ and $|\bar{\alpha}|<N(I)$ where $\bar{\alpha}$ is the algebraic conjugate of α.

Proof of the following facts can be found in [14]–[16].

Theorem 1.1. (a) *If I is a reduced ideal then $N(I)<\sqrt{\Delta}$.*

(b) *If I is a primitive ideal and $N(I)<\sqrt{\Delta}/2$ then I is reduced.*

Let $I=[N(I), b+w_d]$ be primitive then the expansion of $(b+w_d)/N(I)$ as a continued fraction $\langle a_0, \overline{a_1, a_2, \cdots, a_k}\rangle$ of period length k and the sequences of integers P_i, Q_i, $i\geq 0$ are obtained recursively as follows:

$$(P_0, Q_0)=(\sigma b+\sigma-1, \sigma N(I)), \qquad P_{i+1}=a_iQ_i-P_i$$

where $a_i=\lfloor(P_i+\sqrt{d})/Q_i\rfloor$ with $\lfloor\ \rfloor$ being the greatest integer function, and $d=P_{i+1}^2+Q_iQ_{i+1}$.

Let $I=[N(I), b+w_d]$ primitive and reduced. Then the expansion of $(b+w_d)/N(I)$ into a continued fraction yields *all* of the reduced ideals in $\mathcal{O}_K$ equivalent to I; i.e. $I_1=[Q_0/\sigma, (P_0+\sqrt{d})/\sigma]=I\sim I_2=[Q_1/\sigma, (P_1+\sqrt{d})/\sigma]$