3. *Sums of a Certain Class of q-series*

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M. Vowe and H.-J. Seiffert [6] evaluated the sum:

$$\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\frac{1}{2^k(n+k+1)}=\frac{2^n(n-1)!\,n!}{(2n)!}-\frac{2^{-n}}{n} \qquad (n\in N=\{1,2,3,\cdots\}) \tag{1}$$

by identifying it with an Eulerian integral. Subsequently, in our attempt in [4] to find the sum (1), *without* considering this Eulerian integral, we were led naturally to numerous interesting generalizations of (1) obtainable as useful consequences of Kummer's summation theorem [3, p. 134, Theorem 3] in the theory of the familiar (Gaussian) hypergeometric series (see [4] for details). The object of the present note is to derive certain basic (or q-) extensions of (1) and of its various generalizations given already by us [4].

For real or complex q, $|q|<1$, let

$$(\lambda;q)_0=1;\ (\lambda;q)_k=(1-\lambda)(1-\lambda q)\cdots(1-\lambda q^{k-1}),\ \forall k\in N, \tag{2}$$

and

$$(\lambda;q)_\infty=\lim_{k\to\infty}(\lambda;q)_k=\prod_{j=0}^{\infty}(1-\lambda q^j) \tag{3}$$

for an arbitrary (real or complex) parameter λ. Then a q-extension of Kummer's summation theorem [3, p. 134, Theorem 3], employed in our earlier work [4], can be written in the form (cf. [1, p. 526, Equation (1.9)]):

$$\sum_{k=0}^{\infty}q^{k(k-1)/2}\frac{(a;q)_k(q/a;q)_k}{(c;q)_k}\frac{c^k}{(q^2;q^2)_k}=\frac{(ca;q^2)_\infty(cq/a;q^2)_\infty}{(c;q)_\infty}, \tag{4}$$

or, equivalently,

$${}_2\Phi_2\left[\begin{matrix}a,\ q/a;\\ c,\ -q;\end{matrix}\ q,\ -c\right]=\frac{(ca;q^2)_\infty(cq/a;q^2)_\infty}{(c;q)_\infty} \tag{5}$$

in terms of a basic (or q-) hypergeometric ${}_r\Phi_s$ function (cf., e.g., [5, p. 347, Equation (272)]).

Defining the basic (or q-) binomial coefficient by

$$\begin{bmatrix}\lambda\\0\end{bmatrix}=1;\quad \begin{bmatrix}\lambda\\k\end{bmatrix}=(-1)^kq^{k(2\lambda-k+1)/2}\frac{(q^{-\lambda};q)_k}{(q;q)_k},\qquad k\in N, \tag{6}$$

it is easily verified that

$$\begin{bmatrix}\lambda+k-1\\k\end{bmatrix}=\frac{(q^\lambda;q)_k}{(q;q)_k}\qquad (k\in N_0=N\cup\{0\}) \tag{7}$$

and that

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