# 93. *Matrix Prime Number Theorems*

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0. The purpose of this paper is to show how some information on prime numbers in very short intervals, of type $[x, x+x^{\varepsilon}]$ with $\varepsilon>0$ arbitrary, may be derived by classical methods of analytic number theory. We have not attempted in this note to give the sharpest result possible in this direction. For comparison, the known applications for zero-density results give

$$\pi(x+x^{\delta})-\pi(x)\sim x^{\delta}/\log x \quad (\delta>7/12)$$

and a corresponding result for almost all intervals for $\delta>1/6$ (Huxley [1], [3], p. 19).

1. In this section we derive the main estimate in our work. Deductions from it will be given in section 2.

We shall need from the theory the following known results. We use the notation $\rho=\beta+i\gamma$ for the real and imaginary parts of a non-trivial zero of Riemann zeta-function.

(A) The number of $\rho$ with $T\leqslant|\gamma|<T+1$ is $O(\log T)$, where multiplicities are counted.

This is Theorem 9.2 in Titchmarsh [4].

(B) We have $1-\beta\gg(\log\gamma)^{-2/3}(\log\log\gamma)^{-1/3}$.

This is the Vinogradov-Korobov bound ([4], p. 135).

(C) The explicit formula with remainder of prime number theory may be taken as

$$y-\psi(y)=\sum_{|\gamma|<T}\frac{y^{\rho}}{\rho}+O(\log y+yT^{-1}(\log T)^{2})$$

provided $T$, $y$ are greater than 1 and $y$ is bounded by a fixed power of $T$.

This follows from Theorem 3.8 of Patterson [2].

**Notation.** *Let $\eta$ be fixed with $0<\eta<1$, let $X>1$ and $N=X^{\eta}$. Define $\alpha=X^{1/N}$.*

**Theorem.** *Suppose $N$ is an integer. Define $f(\theta)$ to be*

$$\sum_{j=1}^{N}\Big(\sum_{\alpha^{j-1}<m\leqslant\alpha^{j}}\Lambda(m)-\alpha^{j-1}(\alpha-1)\Big)e^{ij\theta}.$$

*Then $f(\theta)$ is $o(X)$ (with constant independent of $\theta$).*

**Remarks.** The integrality of $N$ is assumed to simplify the notation; sums over $m$ in $(\alpha^{-r}X, \alpha^{-r+1}X]$ may be dealt with in the same way in general. The proof shows that the bound may be taken as

$$O(X\exp(-A(\log X)^{1/3}(\mathrm{loglog}\, X)^{-1/3})).$$

We prove first the following