86. *Weyl's Type Criterion for General Distribution Mod 1*

By Masumi NAKAJIMA*) and Yukio OHKUBO**)

(Communicated by Shokichi IYANAGA, M. J. A., Nov. 13, 1989)

1. In 1916, Weyl [6] has proposed the following necessary and sufficient condition for uniform distribution mod 1 of the real number sequences which is now called *Weyl's criterion*: *The sequence* (x_n), $(n=1,2,3,\cdots)$ *is uniformly distributed* mod 1, *if and only if*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i\nu x_n}=0$$

for any natural number ν.

Schoenberg [5], in 1928, first generalized the concept of uniform distribution mod 1 to that of general distribution mod 1 (or asymptotic distribution mod 1) and obtained many interesting results including generalizations of Weyl's criterion. Later many mathematicians have also proposed various forms of criteria for general distribution mod 1 (cf. [3]). But their criteria were not of exponential sum type. Among them, Helmberg [2] has made an interesting contribution to this field with view points concerning mainly numerical computations of the integrals of type: $\int_0^\infty f(x)dx$. Recently the first author found a natural generalization of Weyl's criterion (Theorem 1) which seems new to the authors. In this note, we give this criterion for generally distributed mod 1 sequences with some applications: estimations of some trigonometric sums, a generalization of Erdös-Turán's theorem and a generalization of LeVeque's inequality. The proof of these results and other results in various directions will be given elsewhere.

2. Definition. Let $\mu(x)$ be a distribution function with $\mu(0)=0$, $\mu(1)=1$, $d\mu(x)=w(x)dx$ where $w(x)$ is the density function of $\mu(x)$ satisfying the following conditions: (a) $w(x)$ is piecewise continuous on $[0,1]$ and $0<w(x)<+\infty$. (b) The number of the discontinuity points of $w(x)$ is finite. (c) ${}^{\#}\{z\in[0,1]\,|\,\lim_{x\to z-0}w(x)=0 \text{ or } \lim_{x\to z+0}w(x)=0\}<\infty$, where ${}^{\#}A$ denotes the number of elements of the set A. Then, the real number sequence $(x_n)_{n=1}^{\infty}$ is called *μ-distributed* mod 1, if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi([0,x):\{x_n\})=\mu(x)$$

for each $x\in[0,1]$, where $\chi([0,x):u)$ is the indicator function of $[0,x)$ and $\{x\}$ denotes the fractional part of x.

3. Results. Then we have the main theorem.

Theorem 1. *The sequence* (x_n) *is μ-distributed* mod 1, *if and only if*

*) Department of Mathematics, Rikkyo University.
**) Yakuendai Senior High School.