44. On the Inverse Scattering on the Line and the Darboux Transformation

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In this paper we study the inverse scattering problem for the 1-dimensional Schrödinger operator

$$H(u)=-\frac{d^2}{dx^2}+u(x), \qquad -\infty<x<\infty$$

by the method of the Darboux transformation. Here we assume that the potential $u(x)$ belongs to

$$L_{1,\lambda}=\left\{u\,|\,\text{real valued, continuous and } \int_{-\infty}^{\infty}|x|^{\lambda}|u(x)|\,dx<\infty\right\}$$

for some $\lambda\geqq 0$. In this article, we omitted the proof. See [3] and [4] for details.

1. Jost solutions. Let $f_{\pm}(x,\xi\,;u)$ be the solutions of the eigenvalue problem

$$H(u)f_{\pm}=-f''_{\pm}+u(x)f_{\pm}=\xi^2 f_{\pm}$$

such that $f_{\pm}(x,\xi\,;u)$ behave like $e^{\pm i\xi x}$ as $x\to\pm\infty$ respectively, which are called the Jost solutions, if they exist. If $u(x)\in L_{1,0}$, then $f_{\pm}(x,\xi\,;u)$ exist for $\xi\in \boldsymbol{R}\backslash\{0\}$. Moreover, if $u(x)\in L_{1,1}$, then $f_{\pm}(x,\xi\,;u)$ extended analytically into the complex upper half plane $\operatorname{Im}\xi>0$. More precisely, $e^{\mp i\xi x}f_{\pm}(x,\xi\,;u)-1$ belong to the Hardy space H^{2+} of the upper half plane and, therefore, they admit the integral representation

$$e^{\mp i\xi x}f_{\pm}(x,\xi\,;u)=1\pm\int_0^{\pm\infty}B_{\pm}(x,y)e^{\pm i\xi y}dy. \tag{1}$$

In particular, $f_{\pm}(x,0\,;u)$ are defined. The entries of the S-matrix of $H(u)$ are represented explicitly in terms of the Jost solutions. For example, we have

$$r_{\pm}(\xi\,;u)=\frac{[f_{+}(x,\mp\xi\,;u),f_{-}(x,\pm\xi\,;u)]}{[f_{-}(x,\xi\,;u),f_{+}(x,\xi\,;u)]},$$

where $r_{+}(\xi\,;u)$ and $r_{-}(\xi\,;u)$ are the right and left reflection coefficients respectively, and $[f,g]=fg'-gf'$ is the Wronskian. We refer to [1] for explicit representations of another entries and further information about the scattering data.

2. Levinson's theorem. The following, which is called Levinson's theorem usually, is well known.

Theorem 1 (cf. [1; p. 208]). *A potential $u(x)$ in $L_{1,1}$ without bound states is determined by its right reflection coefficient.*