44. Class Number One Criteria for Real Quadratic Fields. II

By R. A. MOLLIN
Mathematics Department, University of Calgary,
Calgary, Alberta, Canada, T2N 1N4

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This paper continues the work begun in [6]. Therein we gave criteria for real quadratic fields of narrow Richaud-Degert (R-D) type to have class number one. This was a consequence of more general criteria given for real quadratic fields $Q(\sqrt{n})$ with $n\equiv 1 \pmod 4$.

Herein we will deal with positive square-free integers n of wide (R-D) type; i.e., $n=m^2+r$ where r divides $4m$ and $r\in(-m, m]$ with $|r|\neq 1, 4$. The first result generalizes results in [1], [3], [4], [9] and [11].

Theorem 1. *Let $n=l^2+r>7$ be of wide R-D type such that $n\not\equiv 1 \pmod 4$. If $h(n)=1$ then:*

(1) $|r|=2$.

(2) *p is inert in $Q(\sqrt{n})$ for all odd primes p dividing l.*

(3) *If $r=2$ then $l\equiv 0 \pmod 3$.*

(4) *If $r=-2$ then $l\not\equiv 0 \pmod 3$.*

Proof. Since $n\not\equiv 1 \pmod 4$ then 2 is ramified in $Q(\sqrt{n})$. Therefore, there are integers x and y such that $x^2-ny^2=\pm 2$. By [5, Theorem 1.1] $2\geq|r|$; where $|r|=2$ since $|r|\neq 1$ by hypothesis. This secures (1). If p is an odd prime dividing l such that p is not inert in $Q(\sqrt{n})$ then there are integers u and v such that $u^2-nv^2=\pm p$. By [5, Theorem 1.2] $n=7$ and $p=3$ are forced. This secures (2).

If 3 is not inert in $Q(\sqrt{n})$ then $x^2-ny^2=\pm 3$ for some integers x and y. Assume that $x>0$ and that $y>0$ is the *least* positive solution. Thus we may invoke [7, Theorem 108–108a, pp. 205–207] to get that if $x^2-ny^2=3$ then; for $x_1=(2l^2+r)/|r|$ and $y_1=2l/|r|$ (see [2] and [8]):

(i) $0\leq y\leq y_1\sqrt{3}/\sqrt{2(x_1+1)}$

and if $x^2-ny^2=-3$ then:

(ii) $0<y\leq y_1\sqrt{3}/\sqrt{2(x_1-1)}$.

A tedious check shows that $y=1$.

Therefore $x^2-n=\pm 3$; i.e., $x^2-l^2=r\pm 3$. An easy check shows that the only possible solutions to the latter equation occur when either $l=r=2$ or $l=3$, and $r=-2$. Thus, if $n>6$ when $r=2$, and $n>7$ when $r=-2$ then 3 is inert in $Q(\sqrt{n})$; whence $n\equiv 2 \pmod 3$. Therefore, $l\equiv 0 \pmod 3$ if $r=2$, and $l\not\equiv 0 \pmod 3$ if $r=-2$. This secures (3), (4) and the theorem. Q.E.D.

Remark 1. The converse of Theorem 1 is false. For example, if $n=12^2+2=146$ then Theorem 1 (1)–(3) are satisfied, but $h(n)=2$.