8. *Special Values of Euler Products and Hardy-Littlewood Constants*

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(Communicated by Kunihiko KODAIRA, M. J. A., Jan. 13, 1986)

We note an interpretation of Hardy-Littlewood constants (originally constructed by Hardy-Littlewood [2] and generally by Bateman-Horn [1]) as special values of certain Euler products treated in author's papers [3], [4], [5] supplementing the ending remark of [6]. The contents of this report were presented in a seminar of Research Center for Advanced Mathematics on "spatial zeta functions" at Nagoya University in November 1985. The author is grateful to thank Professor T. Sunada for his invitation to that seminar with supplying important information containing vast results of his school partially summarized in Sunada [9], and would like to express hearty thanks to Professor K. Shiga for making that opportunity.

Let $f(X)\in \boldsymbol{Z}[X]$ be a separable primitive polynomial in one variable with coefficients in the rational integers $\boldsymbol{Z}$. For each prime number p we put $N(p, f)=\sharp\{x\in \boldsymbol{F}_p ; \bar{f}(x)=0\}$, where $\bar{f}(X)\in \boldsymbol{F}_p[X]$ denotes the reduction of $f(X)$ modulo p, $\boldsymbol{F}_p$ being the finite field of p elements, and $\sharp$ denotes the cardinality, so $0\leqq N(p, f)\leqq p$. We define the "zeta function" $Z(s, f)$ of f by

$$Z(s, f)=\prod_p (1-N(p, f)p^{-s})$$

where p runs over all prime numbers. (We do not take the inverse since the above form is suitable for our purpose below.) Let $M(f)=\mathrm{Spec}\,(\boldsymbol{Z}[X]/(f))$ be the scheme associated with f over $\mathrm{Spec}\,(\boldsymbol{Z})$, then the Hasse-Weil zeta function $\zeta(s, M(f))$ of the one dimensional space $M(f)$ is equal to

$$\zeta(s, M(f))=\prod_p (1-N(p, f)p^{-s}+\cdots)^{-1},$$

so we can consider $Z(s, f)$ as a truncated zeta function of $M(f)$. Now the Hardy-Littlewood constant $C(f)$ of $f(X)$ is defined via

$$C(f)=\prod_p (1-N(p, f)p^{-1})(1-p^{-1})^{-r(f)}$$

where $r(f)$ denotes the number of irreducible factors of $f(X)$ in $\boldsymbol{Z}[X]$, and p runs over all prime numbers according to the natural order $2, 3, \cdots$ (since this infinite product does not converge absolutely in general).

Theorem A1. *Let $f(X)$ be as above.*

(1) *$Z(s, f)$ is meromorphic in $\mathrm{Re}\,(s)>0$. It is meromorphic on $\boldsymbol{C}$ if $\deg (f)\leqq 1$, and otherwise it has the natural boundary $\mathrm{Re}\,(s)=0$.*

(2) *$Z(s, f)$ is holomorphic in $\mathrm{Re}\,(s)\geqq 1$ and has the following Taylor expansion at $s=1$:*

$$Z(s, f)=C(f)(s-1)^{r(f)}+(\textit{higher order terms}).$$