81. On the Existence and Uniqueness of SDE Describing an n-particle System Interacting via a Singular Potential

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(Communicated by Kôsaku YOSIDA, M. J. A., Nov. 12, 1985)

1. Introduction. Let $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ be a filtered probability space and let $[B^1(t), \cdots, B^n(t)]$ be an $\mathcal{F}_t - BM_0^{2nd}$ ($B^i(t)$ is $\mathbf{R}^{2d}$-valued), where BM_x^r denotes an r-dim. Brownian motion starting from $x \in \mathbf{R}^r$. Let $\mathcal{N}$ denote the set of points $(z^1, \cdots, z^n) \in \mathbf{R}^{2nd}$ such that $z^i = z^j$ for some $i \neq j$ and $z^\perp$ denote $(y, -x)$ for $z = (x, y) \in \mathbf{R}^{2d}$. We consider the following stochastic differential equation (abbreviated: *SDE*) *describing an interacting n-particle system* in $\mathbf{R}^{2d}$ starting from $(z^1, \cdots, z^n) \notin \mathcal{N}$:

(1) $$dZ^i(t) = dB^i(t) + \textstyle\sum_{j: j \neq i} \gamma_j \nabla^\perp H(Z^i(t) - Z^j(t))dt \qquad i = 1, \cdots, n,$$
$$Z^i(0) = z^i \qquad i = 1, \cdots, n,$$

in which,

$$\gamma_i \in \mathbf{R}^1, \ \neq 0 \qquad i = 1, \cdots, n,$$
$$H(z) = g(|z|), \quad (\nabla^\perp H)(z) = (\nabla H(z))^\perp \quad z \in \mathbf{R}^{2d}, \ \neq 0,$$

where $g \in C^2(0, \infty)$ and $\nabla H = (\partial H/\partial z_1, \cdots, \partial H/\partial z_{2d}) \in \mathbf{R}^{2d}$. For a typical example, if we set $g(r) = -(1/2\pi) \log r$ and $d = 1$, then the above system of *SDE* describes a dynamics of n vortices in incompressible and viscous fluid in $\mathbf{R}^2$, where the constants γ_i denote the vorticity of the i-th vortex ([1], [3]). Hence we call this the *SDE representing the vortex flow.* (1) is significant in connection with the nonlinear *SDE* in $\mathbf{R}^{2d}$:

$$dZ(t) = dB(t) + \int_{\mathbf{R}^{2d}} \nabla^\perp H(Z(t) - z)\mu_t(dz)dt,$$

where $B(t)$ is a BM_0^{2d} and $\mu_t(dz)$ is the law of $Z(t)$. Particularly the *SDE representing the vortex flow* is related to the *Navier-Stokes equation* ([3]).

The problem we consider is the existence and uniqueness of a solution of (1). In fact H. Osada ([4]) proved that in the vortex flow case, (1) has a unique strong solution, using an estimate of the fundamental solution of a parabolic equation with a *generalized divergence form.* In this paper, under a suitable condition on the singularity of $g(r)$ at $r = 0$ and assuming that $\{\gamma_i\}$ has the same sign, we prove the unique existence of a solution for a general (1) including the vortex flow case by a *probabilistic* method, which seems simpler than Osada's. But in Osada's argument, the equi-sign property of $\{\gamma_i\}$ is not necessary.

One can explain intuitively the reason why the equi-sign property of $\{\gamma_i\}$ simplifies the situation: Assuming $g'(r) > 0$, we can see that the drift acts on $\{Z^i, Z^j\}$ as if Z^i and Z^j rotate around $(Z^i + Z^j)/2$ clockwise with intensities $\gamma_j g'(r)$ and $\gamma_i g'(r)$ ($r = |Z^i - Z^j|$) respectively. This fact prevents