79. On q-Additive Functions. I

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1. Let q be an arbitrary fixed natural number $\geqq 2$. Then a natural number n can be written in the unique way:

$$n=\sum_{k=0}^{\infty} a_k(n)q^k, \quad 0 \leqslant a_k(n) \leqslant q-1 \quad (q\text{-adic expansion of } n).$$

We say that an arithmetic function $g(n)$ is *q-additive*, if

$$(1) \qquad g(0)=0 \quad \text{and} \quad g(n)=\sum_{k=0}^{\infty} g(a_k(n)q^k)$$

whenever $n=\sum_{k=0}^{\infty} a_k(n)q^k$ (cf. Gelfond [1]).***) The function "Sum of digits" $S_q(n)$ defined by $S_q(n)=\sum_{k=0}^{\infty} a_k(n)$, is a typical example of a q-additive function.

Let $[x]$ denote the integral part of x, and $\zeta(s, r/q)$, $1 \leqslant r \leqslant q$ the Hurwitz zeta function defined by $\zeta(s, r/q)=\sum_{m=0}^{\infty}(m+r/q)^{-s}$ for Re $(s) >1$. We put

$$\mathcal{A}=\Big\{g(n): q\text{-additive function such that the convergence abscissa of } \int_1^{\infty} g([t])t^{-s-1}dt<\infty\Big\},$$

$$\mathcal{B}=\{H(z): \text{Taylor series in } z \text{ with positive radius of convergence}\}.$$

In this article we give a result concerning a relation between $\mathcal{A}$ and $\mathcal{B}$. Our theorem is:

Theorem. *For q given functions $H_r(z) \in \mathcal{B}$, $1 \leqslant r \leqslant q$, there exist a unique $g(n) \in \mathcal{A}$ and a unique $H(z) \in \mathcal{B}$ such that*

$$(2) \qquad \sum_{r=1}^{q} H_r(q^{-s})\zeta\Big(s, \frac{r}{q}\Big)=s\cdot q^s \cdot \int_1^{\infty} g([t])t^{-s-1}dt+q^{s-1}H(q^{-s})\zeta(s).$$

Conversely, for a given $g(n) \in \mathcal{A}$ and an $H(z) \in \mathcal{B}$, there exists a unique system $H_r(z) \in \mathcal{B}$, $1 \leqslant r \leqslant q$, which satisfies (2).

We intend to give, as an application of this result, an explicit summation formula $\sum_{n \leqslant x} g(n)$ for some q-additive functions, in a subsequent article.

2. The following lemma plays an important part in the proof of our Theorem.

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***) The values of g on the set $\{rq^k: 1 \leqslant r \leqslant q-1,\ k \in N\}$, determine completely the q-additive function $g(n)$.