58. On τ Functions of a Class of Painlevé Type Equations. I*)

By Yasuko MÔRI
Department of Mathematics, Ryukyu University

(Communicated by Kôsaku YOSIDA, M. J. A., June 12, 1980)

1. The aim of the present note is to give the description of monodromy preserving deformation of a linear ordinary differential equation of the form

$$\mathcal{L}Y \equiv \left(x\frac{d}{dx}+L\frac{d}{dx}+Mx+N\right)Y=0 \tag{1}$$

in a Hamiltonian form and to establish transformation formulas of the associated 'τ functions' ([2]–[5]). Here the coefficients L, M and N are constant matrices of size r while Y can be a column vector as well as a square matrix of size r of functions of x. We assume that L (resp. M) has distinct eigenvalues which we write $-a_j$ (resp. $-c_j$), $j=1, \cdots, r$ so that $-L$ (resp. $-M$) is conjugate to the diagonal matrix $A=(a_j\delta_{jk})_{j,k=1,\cdots,r}$ (resp. $C=(c_j\delta_{jk})_{j,k=1,\cdots,r}$). Hereafter we shall normalize $-L=QAQ^{-1}$, $-M=C$ so that we can write

$$\mathcal{L}=Q(x-A)Q^{-1}\left(\frac{d}{dx}-C\right)-B=\left(\frac{d}{dx}-C\right)Q(x-A)Q^{-1}-B' \tag{2}$$

by setting $B=LM-N$, $B'=1+ML-N$. We have

$$B'=1+B-[QAQ^{-1}, C]. \tag{3}$$

We also set: $P=Q^{-1}B$, $E_j=(\delta_{kj}\delta_{k'j})_{k,k'=1,\cdots,r}$, and $B_j=QE_jP$. By writing our equation, $\mathcal{L}Y=0$, as

$$\frac{d}{dx}Y=(Q(x-A)^{-1}P+C)Y \tag{4}$$

and observing $(x-A)^{-1}=\sum_{j=1}^{r}(x-a_j)^{-1}E_j$, we see that (1) is equivalent to

$$\frac{d}{dx}Y=\left(\sum_{j=1}^{r}\frac{B_j}{x-a_j}+C\right)Y, \qquad \text{with } B_j \text{ of rank} \leq 1, \tag{5}$$

an equation with regular singularities at $x=a_1, \cdots, a_r$ and an irregular singularity of rank 1 at $x=\infty$. Note that the number of regular singularities is equal to the size r.

Conversely, suppose we are given an equation (5) with rank of $B_j \leq 1$ and $C=(c_j\delta_{jk})$ diagonal. Set $\lambda_j=\operatorname{trace} B_j$ which is an eigenvalue of B_j, and define Q to be the matrix whose j-th column vector $[Q]_j$ is the eigenvector of B_j belonging to the eigenvalue λ_j: $B_j[Q]_j=\lambda_j[Q]_j$.

*) This work was done while the author stayed at RIMS, Kyoto University on leave of absence.