# 82. On the Zero-Free Region of Dirichlet's L-Functions

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1. Let $L(s,\chi)$ $(s=\sigma+it)$ be the Dirichlet $L$-function for a Dirichlet character $\chi$. We denote by $\mathscr{Z}(T)$ the set of all zeros in the region $0<\sigma<1$, $|t|\leqq T$ of all primitive $L$-functions of modulus $\leqq T$. Then the fundamental result on the zero-free region for $L(s,\chi)$ is

**Theorem.** *For any* $\rho\in\mathscr{Z}(T)$ *we have*

$$\mathrm{Re}\,\rho\leqq 1-c_0(\log T)^{-1}, \tag{1}$$

*save for at most one zero, where* $c_0$ *is an effectively computable positive constant. This (possibly existing) exceptional zero* $\beta_1$ *is real and simple, and comes from* $L(s,\chi_1)$ *with a unique real character* $\chi_1$. *Further there exists a function* $c(\varepsilon)>0$ *such that for any* $\varepsilon>0$

$$\beta_1\leqq 1-c(\varepsilon)T^{-\varepsilon}. \tag{2}$$

(1) is the Page-Landau theorem, and (2) is Siegel's theorem in which $c(\varepsilon)$ is not effectively computable. The purpose of the present note is to modify the argument of our preceding note [1] so as to prove this theorem without appealing to the deep function-theoretical properties of $L(s,\chi)$. The details will appear elsewhere.

2. In what follows we assume always that $T$ is sufficiently large.

**Lemma 1.** *Uniformly for* $0\leqq\sigma\leqq 1$ *and for* $\chi(\bmod q)$ *we have*

$$L(s,\chi)\ll(q(|t|+1))^{1-\sigma}\log(q(|t|+2)).$$

*If* $\chi$ *is principal, the region* $|s-1|\leqq 1/2$ *should be excluded.*

**Lemma 2.** *For any* $\rho\in\mathscr{Z}(T)$ *we have*

$$\mathrm{Re}\,\rho\leqq 1-T^{-3}.$$

Lemma 1 is not the best among results of this type, but the above assertion can be proved only by the partial summation. Lemma 2 is quite rough, but this is important in our procedure. To prove it let $L(\rho,\chi)=0$. Either if $\chi$ is complex or if $|\mathrm{Im}(\rho)|\geqq T^{-2}$, then the argument of [2, pp. 43–44] does work also for $L(s,\chi)$. So in these cases we have $\mathrm{Re}\,\rho\leqq 1-T^{-3}$. Otherwise let $a(n)=\sum_{d|n}\chi(d)$. Then $a(n)\geqq 0$ and $a(n^2)\geqq 1$. So by Lemma 1, we have

$$N^{1/2}\ll\sum_{n\leqq N}a(n)(\log N/n)^2$$
$$=2L(1,\chi)N+O(T(\log T)^2).$$

Hence $L(1,\chi)\gg T^{-1}(\log T)^{-2}$, from which the desired assertion follows easily.