41. On the Logarithmic Kodaira Dimension of the Complement of a Curve in P^2

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1. The logarithmic Kodaira dimension introduced by S. Iitaka [1] plays an important role in the study of non-compact algebraic varieties. In this note we calculate the logarithmic Kodaira dimension $\bar{\kappa}(\boldsymbol{P}^2-C)$ of the complement of an irreducible curve C in the complex projective space $\boldsymbol{P}^2$ of dimension 2. We denote by $g(C)$ the genus of the non-singular model of C. In this note, a locally irreducible singular point of C will be called cusp. Our results are as follows:

Theorem. *Let C be an irreducible curve of degree n in $\boldsymbol{P}^2$.*

(I) *If $g(C)\geqslant 1$ and $n\geqslant 4$, then $\bar{\kappa}(\boldsymbol{P}^2-C)=2$.*

(II) *If $g(C)=0$ and C has at least three cusps, then $\bar{\kappa}(\boldsymbol{P}^2-C)=2$.*

(III) *If $g(C)=0$, C has at least two singular points, and one of the singular points is locally reducible, then $\bar{\kappa}(\boldsymbol{P}^2-C)=2$.*

(IV) *If $g(C)=0$ and C has two cusps, then $\bar{\kappa}(\boldsymbol{P}^2-C)\geqslant 0$.*

For the definition of logarithmic Kodaira dimension, see S. Iitaka [1].

Remark 1. It is with ease to show that $\bar{\kappa}(\boldsymbol{P}^2-C)=0$ for any non-singular elliptic curve C of degree 3 in $\boldsymbol{P}^2$.

Remark 2. F. Sakai [5] and S. Iitaka [3], independently of us, showed the same result as Case (I).

2. **Monoidal transformations.** Let

$$\tilde{\boldsymbol{P}}^2=S_t\xrightarrow{\pi_t}S_{t-1}\longrightarrow\cdots\longrightarrow S_1\xrightarrow{\pi_1}\boldsymbol{P}^2$$

be a finite sequence of monoidal transformations with successive centers $p_1,\cdots,p_t$. We pose $\pi=\pi_1\circ\cdots\circ\pi_t:\tilde{\boldsymbol{P}}^2\to\boldsymbol{P}^2$. Let E_i be the exceptional curve of the monoidal transformation π_i. Let us denote by E_i' the proper transform of E_i by $\pi_{i+1}\circ\cdots\circ\pi_t$. By definition, E_i is a divisor in S_i, but we shall use for the sake of simplicity the same letter E_i for $(\pi_{i+1}\circ\cdots\circ\pi_t)^*E_i$ also. Let H be an arbitrary line in $\boldsymbol{P}^2$. We shall use the same letter H for π^*H also.

We frequently use the following lemma to calculate $\bar{\kappa}$.

Lemma. *Let $\pi:\tilde{\boldsymbol{P}}^2\to\boldsymbol{P}^2$, H and E_i be as above. Then we have for any $N\in\boldsymbol{N}$, $n_i\in\boldsymbol{N}\cup\{0\}$ the following:*

$$\dim H^0\Big(\tilde{\boldsymbol{P}}^2,\mathcal{O}\Big(NH-\sum_{i=1}^{t}n_iE_i\Big)\Big)\geqslant\frac{1}{2}(N+1)(N+2)-\sum_{i=1}^{t}\frac{1}{2}n_i(n_i+1).$$