8. *On the Deuring-Heilbronn Phenomenon. II*

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1. Quite recently two simple proofs of the Deuring-Heilbronn phenomenon [4] have been obtained independently by the present author [6] and Jutila [2]. Jutila's proof can be much simplified by appealing to the weight $\Psi_r(n)$ of [6]. But, compared with [2], the real advantage of [6] is in its Lemma 4. To exhibit this, we prove here very briefly a hybrid of two fundamental theorems of Linnik [3] [4] coupled with further simplifications which are embodied in Lemmas 2 and 3 below and which show that whole things are now reduced to a simple application of the Selberg sieve. Similar simplifications are, of course, applicable to the former proofs of Linnik's zero-density theorem [3]. Our new result is as follows:

Theorem. *Let* $1-\delta$ *be the exceptional zero of* $L(s,\chi_1)$, χ_1 *real* (mod q). *And let* $\tilde{N}(\alpha, T, \chi)$ *denote the number of zeros of* $L(s,\chi)L(s+\delta, \chi\chi_1)$ *in the region* $\mathrm{Re}\,(s)\geqq\alpha$, $|\mathrm{Im}\,(s)|\leqq T$. *Then we have, for* $\alpha>3/4$,

$$\sum_{\chi \,(\mathrm{mod}\, q)} \tilde{N}(\alpha, T, \chi) \ll_{\varepsilon} \delta(\log qT)(q^7T^4)^{(1+\varepsilon)((1-\alpha)/(3\alpha-2))}.$$

This may not be the best exponent attainable by our method. A similar but much weaker result can be found in [1; Théorème 14], which was obtained by the power-sum method of Turán. The large sieve extension can be proved quite similarly.

2. In what follows, $B(n)$, $g(r)$, $G(R)$ are all defined in [6].

Lemma 1. *Let*

$$(f^{(1)}\circ f^{(2)})_d=\sum_{[u,v]=d} f^{(1)}_u f^{(2)}_v .$$

Then we have

$$\sum_{d|n} (f^{(1)}\circ f^{(2)})_d=\Big(\sum_{u|n} f^{(1)}_u\Big)\Big(\sum_{v|n} f^{(2)}_v\Big).$$

Lemma 2. *Let* $\eta_d=O(|\mu(d)|\, d^{\varepsilon})$ *and let*

$$F(s,\chi\,;\eta)=\sum_{d=1}^{\infty}\chi(d)d^{-s}\eta_d\prod_{p|d}\Big(1+\frac{\chi_1(p)}{p^{\delta}}-\frac{\chi\chi_1(p)}{p^{1+\delta}}\Big).$$

Then we have, for $\mathrm{Re}\,(s)>1$,

$$\sum_{n=1}^{\infty}\chi(n)B(n)\Big(\sum_{d|n}\eta_d\Big)n^{-s}=L(s,\chi)L(s+\delta,\ \chi\chi_1)F(s,\chi\,;\eta).$$

Lemma 3. *Let*

$$G_d(R)=\sum_{\substack{r\leqq R\\(r,d)=1}}\mu^2(r)g(r),$$