82. The Geometry of Lattices by B-covers

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We have studied certain properties of B-covers in lattices as a generalization of metric betweeness in a normed lattice [2–5]. In this note we shall consider various geometrical properties in lattices by B-covers, B^*-covers and $B^\dagger$-covers.

§1. Preliminaries

We shall use the following definition and lemmas which were obtained in [5].

$B(a, b)=\{x \mid x=(a\frown x)\smile(b\frown x)=(a\smile x)\frown(b\smile x)\ a, b, x\in L\}$ is the B-cover of a and b in a lattice L, and if $c\in B(a, b)$, then we shall write acb.

Lemma 1. axb implies $x\frown(a\smile b)=x=x\smile(a\frown b)$.

Lemma 2. axb implies $a\frown x\geqq a\frown b$, $a\smile b\geqq a\smile x$.

Lemma 3. ax_ib $(i=1, 2)$, ax_1x_2 imply x_1x_2b.

Lemma 4. axb, byc, abc imply xby.

Lemma 5. axb, byc, abc imply $a\frown y\leqq x\frown y$.

Lemma 6. (G) is equivalent to (G*) in a modular lattice,
where (G) $(a\frown c)\smile(b\frown c)=c=(a\smile c)\frown(b\smile c)$,
(G*) $(a\frown c)\smile(b\frown c)=c=c\smile(a\frown b)$.

Lemma 7. If L is modular, then $B(a, b)$ is a sublattice.

Lemma 8. In case L is modular, abc, axb, byc imply axc, ayc.

Lemma 9. In case L is modular, abc, axb, byc imply xyc, axy.

Lemma 10. In order that L be a distributive lattice it is necessary and sufficient that the condition (A) below holds for any elements a, b of L.

(A) $x\in B(a, b)$ if and only if $a\frown b\leqq x\leqq a\smile b$.

Lemma 11. For any elements a, b, c, d of L,

(1) $B(a, b)=B(c, d)$ implies $a\smile b=c\smile d$, $a\frown b=c\frown d$ in any lattice L;

(2) $a\smile b=c\smile d$, $a\frown b=c\frown d$ imply $B(a, b)=B(c, d)$, if and only if L is a distributive lattice.

Lemma 12. In case L is a complemented distributive lattice with $0, I$, then we have $B(a, a')=B(0, I)=L$, where $a\frown a'=0$, $a\smile a'=I$.

§2. Relations between some B-covers

(1) *abc implies $(a\frown b)b(b\frown c)$ and $(a\smile b)b(b\smile c)$.*

(2) *$(a\frown b)b(b\frown c)$ and $(a\smile b)b(b\smile c)$ imply abc.*

(3) *abc implies $a(a\smile b)c$ and $a(a\frown b)c$.*

Proof. Since (1), (2) are easy, we shall prove (3). We have $b=$