107. On Mutant Sets in Semigroups

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Recently, in his paper [2], [3], A. A. Mullin has introduced the notion of mutant sets of general algebraic systems, and contributed the theory of relations in mathematical molecular biology and its related subjects. We are concerned with mutant sets in the Cartesian product of two algebraic systems, especially semigroups. On the fundamental concepts of the Cartesians product, we follow the paper [4] by M. Petrich, and on mutant sets, see references [1], [2] and [3].

Let S and T be semigroups, we define $S\times T$ by the Cartesian product of S and T with coordinate-wise multiplication. Then $S\times T$ is a semigroup.

1) *If A and B are mutant sets of S and T respectively, then $A\times B$ is a mutant set in $S\times T$.*

Proof. By the definition of mutant sets, we have $A^2\subset S-A$, $B^2\subset S-B$. In general, $(A\times B)^2=A^2\times B^2$. Therefore,

$$(A\times B)^2\subset A^2\times B^2\subset (S-A)\times(T-B).$$

Let $a\in S-A$, $b\in T-B$, then $a\overline{\in}A$, $b\overline{\in}B$ and $(a,b)\overline{\in}A\times B$, hence $(a,b)\in S\times T-(A\times B)$. We have

$$(A\times B)^2\subset S\times T-(A\times B).$$

This shows that $A\times B$ is mutant set.

Further, by the definition of multiplication, we have $(a_1,b_1)\cdots(a_n,b_n)=(a_1\cdots a_n, b_1\cdots b_n)$, and it follows that

$$(A\times B)^n=A^n\times B^n. \tag{1}$$

For any subsets A, B and for every positive integer. On the other hand, let $a\in S-A^n$, $b\in T-B^n$, we have $a\overline{\in}A^n$, $b\overline{\in}B^n$, hence $(a,b)\overline{\in}A^n\times B^n$, and by (1), we have $(a,b)\in S\times T-(A^n\times B^n)=S\times T-(A\times B)^n$. Therefore we have the following

Theorem 1. *If A and B are a (m,n)-mutant sets in S and T respectively, the Cartesian product $A\times B$ is a (m,n)-mutant set in $S\times T$.*

Proof. Since A and B are (m,n)-mutant sets in S and T respectively, we have $A^m\subset S-A^n$, $B^m\subset T-B^n$. Then we shall show the relation:

$$(A\times B)^m=A^m\times B^m\subset(S-A^n)\times(T-B^n).$$

Let $a\in S-A^n$, $b\in T-B^n$, we have $a\overline{\in}A^n$, $b\overline{\in}B^n$, and $(a,b)\in A^n\times B^n$. Hence we have