140. *On the Harmonic Summability of Higher Order*

By Hiroshi HIROKAWA

Department of Mathematics, Chiba University, Japan

(Comm. by Kinjirô KUNUGI, M.J.A., Sept. 12, 1967)

§ 1. **Introduction.** Let $\{p_n\}$ be a given sequence of positive numbers and let $P_n=\sum_{k=0}^{n} p_k$. Given a series $\sum_{n=0}^{\infty} a_n$ with its partial sum s_n, if

$$\frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}s_k \longrightarrow s \quad \text{as} \quad n\longrightarrow\infty,$$

the series $\sum_{n=0}^{\infty} a_n$ is said to be summable (N, p_n) to s. A regularity condition of the summability (N, p_n) is

$$p_n/P_n \longrightarrow 0 \quad \text{as} \quad n\longrightarrow\infty.$$

(See Hardy [1, Theorem 16].) The summability $(N, 1/(n+1))$ is known as the harmonic summability. Concerning this summability, Iyengar [2] and Shaney [3] have defined the harmonic summability of higher order, independently. But, their definitions are different. The purpose of this paper is to investigate the relations between these methods of summation. Throughout this paper, p and p' denote positive integers. Iyengar's definition is as follows. Let $\alpha_{n,p}$ define by

$$\sum_{n=0}^{\infty} \alpha_{n,p}x^n \equiv \left\{\frac{1}{x}\log\frac{1}{1-x}\right\}^p = \left\{\sum_{n=0}^{\infty}\frac{x^n}{n+1}\right\}^p.$$

Then the summability (N, p_n) with $p_n=\alpha_{n,p}$ defines the summability (H, p).*) Of course, the summability $(H, 1)$ is the ordinary harmonic summability. If we use Lemma 4 stated below, we see that the regularity condition stated above is satisfied for the summability (H, p). Thus the summability (H, p) is regular. On the other, Shaney's definition is as follows. Let

$$\beta_{n,1}=(n+1)^{-1}$$

and, for $p\geqq 2$,

$$\beta_{n,p}=\left\{(n+1)\prod_{k=1}^{p-1}\log_k(n+\lambda)\right\}^{-1},$$

where $\log_1 x=\log x$, $\log_k x=\log(\log_{k-1} x)$, $k\geqq 2$, and λ is the least positive integer such that $\log_{p-1}\lambda>0$. Then the summability (N, p_n) with $p_n=\beta_{n,p}$ defines the summability (H', p). It should be noted that Shaney has used

---

*) Iyengar [2] has used the notation (N, p) in place of the notation (H, p) and later Varshney [4] has used this one.