139. *A Note on the Powers of Boolean Matrices*

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1. Let $\mathfrak{A}$ be the set of all $n\times n$-matrices each element of which is 1 or 0. For any $A=(a_{ij})$ and $B=(b_{ij})$ of $\mathfrak{A}$, we define multiplication by

$$A\cdot B=\left(\sum_{k=1}^{n}\oplus a_{ik}b_{kj}\right)$$

where $1\oplus 1=1, 1\oplus 0=0\oplus 1=1, 0\oplus 0=0$. It is readily seen that this multiplication is associative and we can consider the m-th power $\underbrace{A\cdot A\cdot \cdots \cdot A}_{m}$ of any element $A\in\mathfrak{A}$. We denote it by A^m. In this paper we shall treat the powers of elements of $\mathfrak{A}$ under this multiplication.

Definitions, Notations, and Preliminary Notes. For any $A=(a_{ij})$ and $B=(b_{ij})$ of $\mathfrak{A}$, we difine operations

$$A\vee B=(a_{ij}\oplus b_{ij}) \text{ and } A\wedge B=(a_{ij}b_{ij}).$$

Then it is easily seen that $\mathfrak{A}$ is a Boolean algebra under these operations. And we can define the ordering $\leqq$ by the usual manner. This definition is equivalent to the proposition that $A\leqq B$ if and only if $a_{ij}=0$ whenever $b_{ij}=0$, and we use also the ordering $<$ defined in such a way that $A<B$ if and only if $A\leqq B$ and $A\neq B$.

E_{st} is the $s\times t$-matrix whose elements are all 1 and O_{st} is the $s\times t$-matrix whose elements are all 0. Particularly if $s=t=n$, we denote them by E and O respectively. Under the above orderings, we can prove that $O\leqq D\leqq E$ for any $D\in\mathfrak{A}$ and that $A\leqq B$ implies $D\cdot A\leqq D\cdot B$ and $A\cdot D\leqq B\cdot D$ for any $D\in\mathfrak{A}$. And $I=(\delta_{ij})$ is the matrix such that $\delta_{ij}=1$ only if $i=j$. For any $A\in\mathfrak{A}$, $I\cdot A=A\cdot I=A$. Further, for each $A\in\mathfrak{A}$, we put $A^k=(a_{ij}^{(k)})$ for each integer $k\geqq 1$. Let $P=(p_{ij})$ be the permutation matrix corresponding to a permutation σ in such a way that only the $p_{i\sigma(i)}$ is 1 in the i-th row and $P^{\top}=(p'_{ij})$ be its transpose. Then P and $P^{\top}$ are the elements of $\mathfrak{A}$ and for each $A\in\mathfrak{A}$ the (i,j)-element of $P\cdot A\cdot P^{\top}$ is

$$\sum_{l=1}^{n}\oplus\left(\sum_{k=1}^{n}\oplus p_{ik}a_{kl}\right)p'_{lj}=\sum_{l=1}^{n}\oplus a_{\sigma(i)l}p_{jl}=a_{\sigma(i)\sigma(j)}.$$

Thus the operation $P\cdot A\cdot P^{\top}$ is equivalent to the operation $PAP^{\top}$ by means of the usual matrix multiplication. In particular, $P\cdot P^{\top}=P^{\top}\cdot P=I$. By virtue of this fact, we can apply the well known theorem for the reducibility of the matrix [1; p 45], and use the