# 40. On the Commutation Relation AB−BA=C

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(Comm. by Kinjirô KUNUGI, M. J. A., Feb. 12, 1971)

We shall deal with commutation relation of the infinitesimal generators of strongly continuous semi-groups on a Banach space $X$.

A few general references for this work are Foias, C., L. Geher and B. Sz.-Nagy [1] and T. Kato [2]. The purpose of this paper is to obtain a generalization of T. Kato's theorem [2]. The proof of the theorem is similar to that of T. Kato's theorem.

The main theorem is as follows.

**Theorem.** *Let $\{e^{sA}\}$and $\{e^{tB}\}$ be two contraction semi-groups on a Banach space $X$ satisfying the relation*

(1) $$e^{sA}e^{tB}=e^{tB}e^{tsC}e^{sA} \qquad 0\leqq s,t<\infty$$

*for some contraction semi-group $\{e^{uC}\}$ and suppose that $D(C)\supset D(B)$. Then*

(a) $\Omega=D(AB)\cap D(BA)$ *is dense in* $X$

(b) $(AB-BA)x=Cx$ *for* $x\in\Omega$

(c) $(A-a)(B-b)\Omega=X$ *for all $a,b$ satisfying* $\mathrm{Re}(a)>0$, $\mathrm{Re}(b)>0$.

*Conversely, let $C$ be the infinitesimal generator of a contraction semi-group. We suppose that $D(C)\supset D(A)$, $D(C)\supset D(B)$ and $C$ commutes with $R(a;A)$ and $R(b;B)$ for some pair $a,b$ satisfying* $\mathrm{Re}(a)>0$, $\mathrm{Re}(b)>0$, *and that there exists a dense linear subset $\Omega$ of $D(AB)\cap D(BA)$ for which* (b) *holds. Furthermore, if we suppose, for some pair $a,b$ satisfying* $\mathrm{Re}(a)>0$, $\mathrm{Re}(b)>0$, *$(A-a)(B-b)\Omega$ is dense in $X$. Then* (1) *holds.*

**Remark.** *If the condition $D(C)\supset D(B)$ of the first part of the theorem is replaced by $D(C)\supset D(A)$, then we have, in* (c), *$(B-b)(A-a)\Omega=X$ for all $a,b$ satisfying* $\mathrm{Re}(a)>0$, $\mathrm{Re}(b)>0$.

**Proof of the first part.** Multiplication of (1) by $e^{-bt}$ followed by an integration with respect to $t$ on $(0,\infty)$ yields

(2) $$e^{sA}(B-b)^{-1}=(B+sC-b)^{-1}e^{sA} \qquad s\geqq 0,$$

whenever $\mathrm{Re}(b)>0$.

Since, for sufficiently small $s>0$, $B+sC$ generates a contraction semi-group by Hille-Yosida's theorem. Differentiation of (2) with respect to $s$ followed by setting $s=0$ leads to

$$A(B-b)^{-1}\supset(B-b)^{-1}A-(B-b)^{-1}C(B-b)^{-1}$$

and hence, for $\mathrm{Re}(a)>0$ and $\mathrm{Re}(b)>0$,