# 65. *Asymptotic Almost Periodicities and Ergodic Theorems.*

By Kôsaku YOSIDA.
Mathematical Institute, Osaka Imperial University.

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1. *Introduction.* Two ergodic theorems, the mean ergodic theorem (M. E. T.) and a generalisation of Fréchet-Kryloff-Bogoliouboff's theorem (F-K-B E. T.), were obtained in the preceding notes.[1] There are some strong difference or gap between these two ergodic theorems. The purpose of the present note is to fulfill this gap with new ergodic theorems and to show that these theorems (including the M. E. T. and the F-K-B E. T.) are intimately related to the properties of *asymptotic almost periodicity*, to be defined below.

Let $T$ denote a continuous (bounded) linear operation defined on a Banach space $B$ to $B$, and consider the sequence $\{T^n \cdot x\}$, $x \in B$, $n = 1, 2, \dots$. Corresponding to the various assumptions of *total boundedness* of $\{T^n \cdot x\}$, we may obtain various ergodic theorems together with the respective properties of asymptotic almost periodicity (in $n$) of $T^n \cdot x$. This simple idea was suggested by Bochner-Neumann's theory[2] (B-N theory) of almost periodic functions in groups. However, since we do not assume the existence of the inverse $T^{-1}$ of $T$, we are here concerned with the *semi-group* of the addition of positive integers. We also remark that a new proof of the existence of the mean for Bohr's (Stepanoff's, Muckenhaupt's and other author's) almost periodic functions may be obtained by virtue of the ergodic theorems. Combined with the Fourier analysis in the B-N theory, the M. E. T. yields a Fourier expansion theorem and a theorem of existence of the proper values for *unitary* (isometric) *operator* $T$ of $B$. Lastly it is to be noted that the B-N theory also suggests us not to confine ourselves to the Banach spaces; the (ergodic) theorems obtained may be extended to *linear topological spaces.*

2. *Ergodic theorems and asymptotic almost periodicities.*

*Theorem 1. We assume that $T$ satisfies the following total boundedness:*

(1) $\quad \| T^n \| \leq$ *a constant* $C$ $(n = 1, 2, \dots)$,

(2) $\quad \begin{cases} \textit{for given } x \in B, \textit{ the sequence } \{x_n\},\ x_n = \dfrac{T + T^2 + \cdots + T^n}{n} \cdot x \\ (n = 1, 2, \dots), \textit{ contains a subsequence weakly convergent to a} \\ \textit{point } x_0 \in B. \end{cases}$

*Then $x_n$ converges strongly to $x_0$ and we have $T \cdot x_0 = x_0$. If* (2) *is satisfied for all $x \in B$, then $x_0$ is defined by a continuous linear operation $T_1$ such that*

1) Proc. **14** (1938), 286–294. Cf. also S. Kakutani: ibd., 295–298.
2) Trans. Amer. Math. Soc. **37** (1935), 21–50.