59. *A Relation between the Theories of Fourier Series and Fourier Transforms.*

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(Comm. by M. FUJIWARA, M.I.A., July 12, 1940.)

1. Let $f(x)$ be defined in $(-\infty, \infty)$ and belong to some class $L_p(p \geqq 1)$. If there exists a function $F(t)$ such that

$$\lim_{A\to\infty}\int_{-\infty}^{\infty}\left|F(t)-\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}f(x)e^{-itx}dx\right|^q dt=0\,,$$

then $F(t)$ is called the Fourier transform of $f(x)$ in L_q. The Titchmarsh theory states that if $f(x)\in L_p$ $(1\leqq p\leqq 2)$, then $f(x)$ has the Fourier transform $F(t)$ in $L_{p'}$ where $1/p+1/p'=1$.

Let $\varphi(x)$ be a periodic function with period $2R\,(R>0)$ and belong to $L_p(-R, R)$ and consider its Fourier series

$$\varphi(x)\sim\sum_{-\infty}^{\infty}c_n e^{-\frac{in\pi}{R}x}\,,\qquad c_n=\frac{1}{2R}\int_{-R}^{R}\varphi(x)e^{-\frac{in\pi}{R}x}dx\,.$$

It is well known that there exist close analogies between the Fourier transforms and Fourier series. The Fourier coefficient c_n corresponds to the Fourier transform. For example the convergence of $\sum|c_n|^a$ stands for the integrability of $|F(t)|^a$ in $(-\infty, \infty)$. Thus the analogy of Hausdorff-Young theorem on Fourier series is Titchmarsh theorem on Fourier transform which asserts that $\int_{-\infty}^{\infty}|F(t)|^{p'}dt<\infty$, if $1<p\leqq 2$.[1]

In this paper I shall prove theorems which make the analogies of this type clearer. The case where $F(t)$ is the Fourier-Stieltjes transform of a probability distribution was discussed recently by the author.[2]

2. *Theorem 1. Suppose that $f(x)\in L_p(-\infty, \infty)$ $(p>1)$ and has the Fourier transform $F(t)$ in $L_q(-\infty, \infty)$ for some $q\,(\geqq 1)$. We define a periodic function $\varphi(t)$ with period $2R$ which concides with $F(t)$ in $(-R, R)$. If c_n is the Fourier coefficient of $\varphi(t)$, then*

$$\sum_{n=-\infty}^{\infty}|c_n|^p\leqq\frac{A_p}{R^{p-1}}\int_{-\infty}^{\infty}|f(x)|^p dx\,,\tag{2.1}$$

where A_p is a constant depending only on p and not of $f(x)$ and R.

Theorem 2. Let $\varphi(t)\in L_1(-R, R)$ and its Fourier series be

$$\varphi(t)\sim\sum_{n=-\infty}^{\infty}c_n e^{\frac{in\pi}{R}t}\,.$$

1) E. C. Titchmarsh, A contribution to the theory of Fourier transforms, Proc. London Math. Soc., **23** (1924), 279–289.

A. Zygnumed, Trigonometrical series, Warszawa, 1935. p. 316.

2) T. Kawata, The Fourier series of the characteristic function of a probability distribution, Tohoku Math. Journ. **47** (1940).