83. *On the Theory of Spectra.*

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The "algebraization" of the spectral theory, inaugurated by J. von Neumann, H. Freudenthal and S. Steen, was taken up recently by S. Kakutani, F. Riesz, M. H. Stone and B. Vulich,[1] and was treated with their respective methods and results. The purpose of the present note is to give a ring-lattice-theoretic treatment of the problem, stressing the analogy to the field of real numbers. Without assuming metrical (even topological) nor divisibility axiom, a *characterisation of the function ring of the Borel-measurable functions*[2] is obtained. Thus the results may be applied to the operator theory as well as to the theory of probability.

§ 1. *Axioms of Pythagorean ring.* A system $\mathfrak{R}$ of elements A, $B, \dots, X, Y, Z$ is called a "Pythagorean ring" if it satisfies the following axioms.

(A-1) $\mathfrak{R}$ is a commutative, associative ring with unit I, admitting the field of real numbers as coefficients (Operatoren).—The real numbers will be denoted by small greak letters.

(A-2) $X^2=0$ implies $X=0$.

(A-3) If non-zero element X of the form $X=Y^2$ is called "positive" (in symbol $X>0$), then the sum of positive element and "non-negative" element is positive, viz. $X^2\neq 0$ or $Y^2\neq 0$ implies the existence of $Z^2\neq 0$ such that $X^2+Y^2=Z^2$.

(A-4) By the semi-order relation $X>Y(X-Y>0)$, there exists, for all X, the lowest upper bound (l. u. b.) $\sup(X, 0)$ of X and 0.—We will write $\sup(X, 0)=X^+$, $\sup(-X, 0)=X^-$ and $|X|=X^++X^-$.

(A-5) $X^+\cdot X^-=0$ for all X.

(A-6) Monotone increasing sequence $\{X_n\}$ bounded from above admits the l. u. b. $\sup\limits_{n\geq 1} X_n$.

(A-7) If $A>0$, $X_i\geqq 0$, $X_{i+1}\geqq X_i$ and $\sup\limits_{i\geq 1} X_i$ exists, then $A\cdot\sup\limits_i X_i=\sup\limits_i (A\cdot X_i)$.

Remark 1. The "real" character of $\mathfrak{R}$ is expressed by the "Pythagorean axiom" (A-3) together with (A-2). (A-4) and (A-6) are lattice-theoretic axioms.[3] (A-5) is equivalent to $|X^2|=|X|^2$, and (A-7) means a generalised distributive law.

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1) J. von Neumann: Rec. Math., **43** (1936), 415-484. H. Freudenthal: Proc. Akad. Amsterdam, **39** (1936), 641-651. S. Steen: Proc. London Math. Soc., **41** (1936), 361-392. S. Kakutani: Proc. **15** (1939), 121-123. F. Riesz: Ann. Math., **41** (1940), 174-206. M. H. Stone: Proc. Nat. Acad. Sci., **26** (1940), 280-283. B. Vulich: C. R. URSS, **26** (1940), 850-859. The last two papers appeared during the preparation of the present note. In the redaction, the writer is much suggested by Steen's paper.

2) Not necessarily bounded!

3) Concerning lattice see G. Birkhoff: Lattice Theory, New York (1940).