105. On the Transformation of a Thetafunction by Siegel's Modular Substitutions.

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The transformation-formula of a theta-function of the 1st degree by Siegel's modular substitution was already discovered.[1] But in point of view of systematic, it is desirable to deduce it along Siegel's line. Our way is not essentially different from the classical one, but somewhat formally simpler.

Let $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a homogeneous modular substitution, that is,

$$(1) \qquad A^*D-C^*B=E, \qquad B^*D=D^*B, \qquad A^*C=C^*A.$$

Then a symmetric matrix Q with positive imaginary part is transformed by the substitution M into a matrix Q_0 of the same kind.

$$Q_0=(AQ+B)\,(CQ+D)^{-1}.$$

We define the thetafunction of the 1st degree with the modulous Q, the argument u, and the characteristics g and h, as follows

$$\vartheta(u;\,g,\,h;\,Q)=\sum_x e^{\pi i(x+g)'Q(x+g)+2\pi i(x+g)'(u+h)},$$

where x runs over the n dimensional lattice points. It is uniformly convergent in u when Q has a positive imaginary part. Now we consider the thetafunction as a function of v, says, $\varphi(v)=\vartheta(u;\,g,\,h;\,Q)$, where $u=(CQ+D)^*v$.

Then the transformation $v\rightarrow\begin{cases} v+\underline{E} \\ v+\underline{Q_0} \end{cases}$ corresponds to the transformation $u\rightarrow\begin{cases} u+\underline{CQ+D} \\ u+\underline{AQ+B} \end{cases}$ respectively, and

$$\vartheta(u+\underline{CQ+D};\,g,\,h;\,Q)=\sum_x e^{\pi i(x+g+\underline{C})'\,Q(x+g+\underline{C})+2\pi i(x+g+\underline{C})'\,(u+h)}$$
$$\times\; e^{-\pi i\underline{C}'Q\underline{C}-2\pi i\underline{C}'u+2\pi i(g'\underline{D}-h'\underline{C})}$$
$$=\vartheta(u;\,g,\,h;\,Q)\,e^{-\pi i\underline{C}'Q\underline{C}-2\pi i\underline{C}'u+2\pi i(g'\underline{D}-h'\underline{C})},$$

$$\vartheta(u+\underline{AQ+D};\,g,\,h;\,Q)$$
$$=\vartheta(u;\,g,\,h;\,Q)\,e^{-\pi i\underline{A}'Q\underline{A}-2\pi i\underline{A}'u+2\pi i(g'\underline{B}-h'\underline{A})}.$$

Small resp. large letters mean vectors resp. matrices of nth. dimension $\overline{P}$ resp. $\underline{P}$ means νth. column resp. raw-vector of matrix P, where $1\leqq\nu\leqq n$ is a given number, and $\underline{\overline{P}}$ the νth. diagonal component of P, while $(\underline{\overline{P}})$ represents the vector whose νth. component is $\underline{\overline{P}}$, $(x=1, 2, \ldots\ldots, n)$. * resp.' represents "transposition" of a matrix resp. vector.

1) A. Krazer. Lehrbuch der Thetafunctionen.