A combinatorial approach to Coxeter groups

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1 Introduction

Let $M = (m_{ij})_{i,j\in I}$ be a Coxeter matrix over a set I. A Coxeter system of type M is a pair (W,S) consisting of a group W and a set $S = \{s_i \mid i \in I\} \subseteq W$ such that $(s_i s_j)^{m_{ij}} = 1_W$ for all $i,j \in I$ and such that the set of these relations yields a presentation of W; hence such that $W \cong \langle S \mid ((s_i s_j)^{m_{ij}})_{i,j\in I}\rangle$ for short. The aim of the present paper is to give a combinatorial proof of the following

Fundamental Fact: Let M be a Coxeter diagram over a set I, let (W,S) be a Coxeter system of type M where $S = \{s_i \mid i \in I\}$. Then the order of the product $s_i s_j$ is equal to m_{ij} for all $i,j \in I$.

This is well known and obtained by a 2-line argument as a very first observation about the geometric representation of a Coxeter group. It seems therefore appropriate to explain why we are nevertheless interested in replacing this short geometric argument by a combinatorial proof which takes about 20 pages. There are in fact two main reasons.

In [7] Tits solved the word problem for Coxeter groups. In that paper he indicated how to use his result in order to produce a 'combinatorial' proof of the classification of finite Coxeter groups. In [3] it is shown, how to employ folding techniques in order to give a short proof of the classification based on Tits' original idea. This gives hence a proof which does not rely on classification of the positive definite bilinear forms associated to Coxeter matrices (which basically corresponds to Coxeter's original proof in [1] and [2]). However, this proof still relies on the geometric representation of a Coxeter group because the fact above is used in [7] for the solution of the word problem. We also mention that in [6] and [8] the theory of Coxeter groups is developed to a large extent in a purely combinatorial set-up. As both references use the geometric representation only to prove the fact above, the question about a

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