The closeness of the range of a probability on a certain system of random events — an elementary proof

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Abstract

An elementary combinatorial method is presented which can be used for proving the closeness of the range of a probability on specific systems, like the set of all linear or affine subsets of a Euclidean space.

The motivation for this note came from the second author's research in statistics: high breakdown point estimation in linear regression. By a probability distribution P, defined on the Borel σ-field of $\mathbb{R}^p$, a collection of regression design points is represented; then, a system $\mathcal{V}$ of Borel subsets of $\mathbb{R}^p$ is considered. Typical examples of $\mathcal{V}$ are, for instance, the system $\mathcal{V}_1$ of all linear, or $\mathcal{V}_2$ of all affine proper subspaces of $\mathbb{R}^p$. The question (of some interest in statistical theory) is:

$$\text{Is there an } E_0 \in \mathcal{V} \text{ such that } P(E_0) = \sup\{P(E)\colon E \in \mathcal{V}\}? \tag{1}$$

For some of $\mathcal{V}$, the existence of a desired E_0 can be established using that (a) $\mathcal{V}$ is compact in an appropriate topology; (b) P is lower semicontinuous with respect to the same topology. The construction of the topology may be sometimes tedious; moreover the method does not work if, possibly, certain parts of $\mathcal{V}$ are omitted, making $\mathcal{V}$ noncompact. Also, a more general problem can be considered:

$$\text{Is the range } \{P(E)\colon E \in \mathcal{V}\} \text{ closed?} \tag{2}$$

The positive answer to (2) implies the positive one to (1). The method outlined by (a) and (b) cannot answer (2) — we have only lower semicontinuity, not full continuity.

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