On the Expansion of a Function in Terms of Spherical Harmonics in Arbitrary Dimensions

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Dedicated to Johann Walter

1 A historical survey

It is well known that a function which is continuously differentiable on the unit circle S^1 can be expressed as a uniformly convergent Fourier series. It is less known that a function which is continuously differentiable on the unit sphere S^2 in $\mathbb{R}^3$ can be expanded in terms of a uniformly convergent series of spherical harmonics, a so–called Laplace series [1] (Kellogg [20,p.259]). Both results can be traced back at least to Dirichlet (1829 and 1837, respectively), although the notion of uniform convergence was brought out a little later through the work of Gudermann (the teacher of Weierstrass), Seidel (a student of Dirichlet), Stokes and Weierstrass. The importance of such an expansion is due to the fact that the solution of the Dirichlet problem for the Laplace equation on the unit disc or unit ball can then be given in terms of a uniformly convergent series of elementary functions. The first edition of Heine's handbook of spherical harmonics which appeared in 1861 and reproduced Dirichlet's 1837 proof does not contain the notion of uniform convergence. The second edition of 1878 does have this notion [14,p.478f.]; at the same time it takes a critical attitude towards Dirichlet's proof [14,p.434]. We shall comment on this proof in Remark 1 at the end of §2.

[1] The classical treatise of Courant–Hilbert [7,p.513] obtains this result as a special case of an expansion of a function in terms of eigenfunctions of a second–order elliptic operator [7,p.369] and is forced to assume that the function be in $C^2(S^2)$. - The three editions of MacRobert's book [24,p.131] contain the assertion that every function in $C^0(S^2)$ has a convergent Laplace series. Experience from Fourier series renders this claim at once highly improbable and it is in fact false (see, for example, [2,p.211]).

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